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Renormalization of noncommutative ϕ^4-theory by multi-scale analysis

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Abstract

In this paper we give a much more efficient proof that the real Euclidean ϕ^4-model on the four-dimensional Moyal plane is renormalizable to all orders. We prove rigorous bounds on the propagator which complete the previous renormalization proof based on renormalization group equations for non-local matrix models. On the other hand, our bounds permit a powerful multi-scale analysis of the resulting ribbon graphs. Here, the dual graphs play a particular rôle because the angular momentum conservation is conveniently represented in the dual picture. Choosing a spanning tree in the dual graph according to the scale attribution, we prove that the summation over the loop angular momenta can be performed at no cost so that the power-counting is reduced to the balance of the number of propagators versus the number of completely inner vertices in subgraphs of the dual graph.

1 Introduction

Field theories on noncommutative spaces became very popular after the discovery that they arise in limiting cases of string theory [1, 2]. Although from string theory's point of view there is no reason that the limit is a well-defined quantum field theory, there has been an enormous activity aiming at renormalization proofs for noncommutative quantum field theories. Most of the attempts focused at the Moyal plane with the associative and noncommutative product

$$(a\star b)(x)=\int\frac{d^4k}{(2\pi)^4}\int d^4y\; a(x+\tfrac{1}{2}\theta\cdot k)\,b(x+y)\,\mathrm{e}^{\mathrm{i}k\cdot y}\,. \tag{1.1}$$

It turned out that the noncommutative analogs of typical field theoretical (in particular four-dimensional) models on the Moyal plane are not renormalizable due to the UV/IR-mixing problem [3]. The construction of dangerous non-planar graphs was made precise in [4] where the problem was traced back to divergences in some of the Hepp sectors which correspond to disconnected ribbon subgraphs wrapping the same handle of a Riemann surface.

Recently, the renormalization of the noncommutative ϕ^4_4-model was achieved [5] within a Wilson-Polchinski renormalization scheme [6, 7] adapted to non-local matrix models [8]. The renormalizable model is defined by the action functional

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{1}{2}\mu_0^2\,\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x) \;, \quad (1.2)$$

where $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu$ and the Euclidean metric is used.

At first sight, the appearance of the translation invariance breaking harmonic oscillator potential for the ϕ^4-action (1.2) might appear strange. However, the renormalization proof shows that there is a marginal interaction which corresponds to that term and as such requires its inclusion in the initial action. Moreover, thanks to the oscillator potential, the action (1.2) becomes invariant under the Langmann-Szabo duality [9] which exchanges position space and momentum space.

We review the main ideas of the renormalization proof, in particular the analysis of ribbon graphs, in Section 2. However, it must be underlined that the proof given in [5] relies on a numerical determination of the asymptotic scaling dimensions of the propagator. Our paper fills this gap by computing rigorous bounds on the propagator, at least for large enough Ω. This will be done in Section 3.

On the other hand, our bounds permit another renormalization strategy which turns out to be much more efficient. See Section 4. The strategy is inspired by constructive methods [10]. The key is a scale decomposition of the propagator and an estimation procedure of the ribbon graphs which takes into account the scale attribution. The proof is carried out for the duals of the ribbon graphs, because the set of independent variables is particularly transparent in dual graphs.

The methods developed in this paper will be crucial to write a constructive version of [8, 5]. Actually the main obstacle to the construction of the usual ϕ^4 model is the non-asymptotic freedom of the theory. In noncommutative $\mathbb{R}^4$, the parameter Ω controls the UV/IR mixing. When it reaches 1, the entanglement is maximum, and the β function vanishes [11]. In this view, the Ω-region close to 1, for which we prove analytical estimates is particularly important.

2 Main ideas of the previous renormalization proof

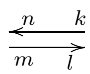
In order to make this paper self-contained, we review the main ideas of the renormalization proof given in [5] for the quantum field theory associated with the action (1.2).

In order to avoid the oscillating phase factors of the $\star$-product in momentum space, the first step is to pass to the matrix base of the Moyal plane, where the action (1.2)

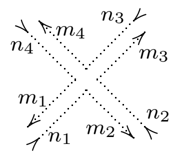
becomes

$$S[\phi] = 4\pi^2\theta_1\theta_2 \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}\Delta_{m,n;k,l}\phi_{mn}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\ . \tag{2.1}$$

As usual, we define the quantum field theory by the partition function, which is expanded into Feynman graphs. As the fields are described by matrices ϕ_{mn}, the resulting Feynman graphs are ribbon graphs build of propagators and vertices,



$= G_{m,n;k,l}\ ,$



$= \delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1}\ .$ (2.2)

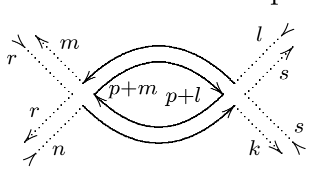
The propagator $G_{mn;kl}$ is the inverse of the kinetic matrix $\Delta_{mn;kl}$ in (2.1). We recall the explicit formula in (3.1) and (3.2). Due to the $SO(2)\times SO(2)$-symmetry of the action, $G_{mn;kl}\neq 0$ only if $m+k=n+l$. Matrix indices which are not determined by this index conservation or as external indices of the graph are summation indices. The corresponding index summation is possibly divergent and requires a regularization.

In [8, 5] the regularization consists in a smooth cut-off of the propagator indices as a function of a renormalization scale Λ,

$$Q_{\substack{m^1\\m^2};\substack{n^1\\n^2};\substack{k^1\\k^2};\substack{l^1\\l^2}}(\Lambda) = \Lambda\frac{\partial}{\partial\Lambda}\left(\prod_{i\in m^1,m^2,\dots,l^1,l^2}\chi\Big(\frac{i}{\theta\Lambda^2}\Big)G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\right)\ , \tag{2.3}$$

where $\chi(x)$ is smooth with $\chi(x)=1$ for $x\leqslant 1$ and $\chi(x)=0$ for $x\geqslant 2$. This implies $Q_{mn;kl}(\Lambda)\neq 0$ only if $\max(m^1,m^2,\dots,l^1,l^2)\in[\theta\Lambda^2,2\theta\Lambda^2]$. The graph is then realized by the differentiated cut-off propagators $Q(\Lambda_i)$ which regulate the index summations. At the end, the nested integral over $\frac{d\Lambda_i}{\Lambda_i}$ is performed within an interval characterized by mixed boundary conditions [7]. Actually, the graphs are build recursively by adding a new propagator. This allows an inductive proof of the power-counting behavior. On the other hand, one has to carefully discuss the location of the valence of the graph where one attaches a leg of the additional propagator. This discussion alone extends over 20 pages in [8].

It is time for an example. We consider the (planar) one-loop four-point graph



$$= \left\{\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda_2}{\Lambda_2}\int_{\Lambda_2}^{\Lambda_0}\frac{d\Lambda_1}{\Lambda_1}\sum_p Q_{m,p+m;p+l,l}(\Lambda_2)\,Q_{n,p+m;p+l,k}(\Lambda_1)\right\} + \{\Lambda_1\leftrightarrow\Lambda_2\}$$
$$+ A_{rm;ls;sk;nr}(\Lambda_R)\ . \tag{2.4}$$

The choice of the boundary conditions is a preliminary one which ensures *convergent integrals at the expense of infinitely many initial data* $A_{rm;ls;sk;nr}(\Lambda_R)$. This will be corrected later in (2.11).

One of the bounds we prove in this paper can be put in the following form

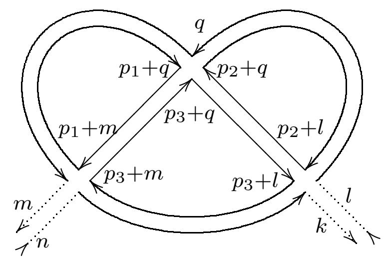
$$G_{{m^1 \atop m^2},{n^1 \atop n^2};{k^1 \atop k^2},{l^1 \atop l^2}} \leqslant K \int_0^1 d\alpha\, e^{-c\alpha(m^1+m^2+n^1+n^2+k^1+k^2+l^1+l^2)} . \tag{2.5}$$

From the remarks made after (2.3) on the range of the maximal index we conclude

$$\Big|Q_{{m^1 \atop m^2},{n^1 \atop n^2};{k^1 \atop k^2},{l^1 \atop l^2}}(\Lambda)\Big| \leqslant 32K \max_x \chi'(x) \int_0^1 d\alpha\; e^{-c\theta\Lambda^2\alpha} \leqslant \frac{32K \max_x \chi'(x)}{c\theta\Lambda^2} . \tag{2.6}$$

We thus estimate the summation over p in (2.4) by the maximum of the propagators Q over p and a volume factor $(2\theta\Lambda_2^2)^2$ from the support of the cut-off function. This shows that the integral (2.4) is estimated by a constant times $\ln\frac{\Lambda}{\Lambda_R}$.

The scaling of (2.6) and the volume of the support of (2.3) with respect to any index seem to suggest that N-point graphs have, as in commutative ϕ^4_4-theory, a power-counting degree $4-N$. However, this conclusion is too early. Namely, there is a problem in presence of completely inner vertices, which require additional index summations. The following graph



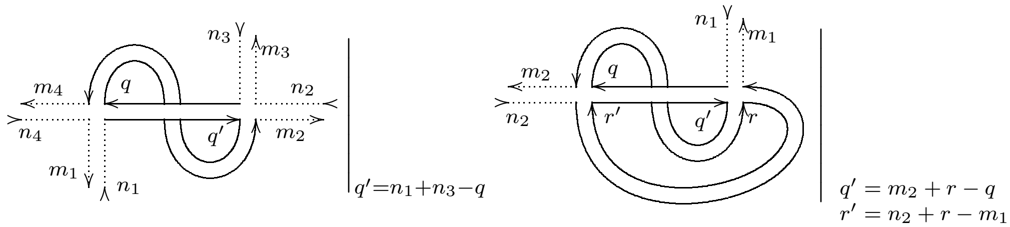
(2.7)

entails *four* independent summation indices p_1, p_2, p_3 and q, whereas for the power-counting degree $4-N$ we should only have three of them. It requires a more careful analysis of the scaling behavior of the propagator to show that the q-summation can actually be performed at no cost, i.e. without a volume factor. The reason is that the propagators show some sort of quasi-locality which implies that the contribution of a propagator $G_{m,n;k,l}$ to a graph is strongly suppressed if $\|m-l\|$ is large. Thus, taking for given m the entire sum over l does not change the power-counting behavior,

$$\left|\max_{m^i}\left(\sum_{l^1,l^2}\max_{n^i,k^i} Q_{{m^1 \atop m^2},{n^1 \atop n^2};{k^1 \atop k^2},{l^1 \atop l^2}}(\Lambda)\right)\right| \leqslant \frac{K'}{\theta\Lambda^2} . \tag{2.8}$$

The two bounds (2.6) and (2.8) together ensure the expected power-counting behavior for all *planar* ribbon graphs. But (2.8) does even more: it ensures the *irrelevance of all*

non-planar graphs. For instance, in the non-planar graphs



(2.9)

the summation over q and q, r, respectively, is controlled by (2.8), i.e. the quasi-locality of the propagator, so that the graphs in (2.9) can be estimated without any volume factor.

We recall from [8] that the non-planarity of ribbon graphs is classified by the number B of boundary components and the genus $g = 1 - \frac{1}{2}(F - I + V)$ of the Riemann surface on which the graph is drawn. Here, V and I are the number of vertices and edges (inner double lines) of the graph. To determine the number F of faces we close the external legs, that is, we connect the outgoing arrow labelled m_i of an external leg directly with its incoming arrow n_i. Then, F is the number of closed single lines and B the number of those closed lines which carry external legs. Then, according to [8, 5], the power-counting degree of a N-leg ribbon graph in four dimensions is

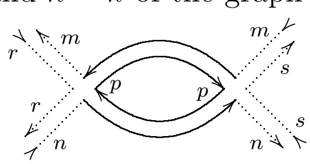
$$\omega = (4 - N) - 4(2g + B - 1) . \tag{2.10}$$

The left graph in (2.9) has topology $B = 2, g = 0$ and the right graph $B = 1, g = 1$.

As a result, there remain only the planar two- and four-leg graphs which can be relevant and marginal. The quasi-locality of the propagator improves the situation in selecting only

- the planar four-leg graphs with *constant index* along the trajectory as marginal,
- the planar two-leg graphs with *constant index* along the trajectory as relevant,
- the planar two-leg graphs with an *accumulated index jump of 2* along the trajectory as marginal.

We refer to [5] for details. The trajectories are the open single lines of the graph (before the closure which identifies the faces). This leaves still an infinite number of divergent graphs. However, there is a discrete Taylor expansion about vanishing external indices which decomposes these divergent graphs into four relevant and marginal base functions and an irrelevant remainder. For instance, the decomposition for the marginal case $m = l$ and $n = k$ of the graph (2.4) reads



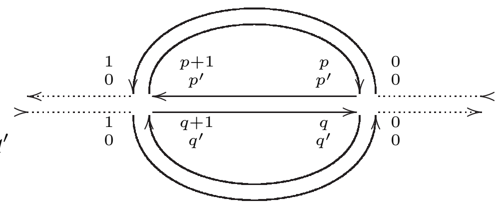
$$= \left\{ \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda_2}{\Lambda_2} \int_{\Lambda_2}^{\Lambda_0} \frac{d\Lambda_1}{\Lambda_1} \sum_p \big(Q_{m,p;p,l}(\Lambda_2)\, Q_{n,p;p,n}(\Lambda_1) - Q_{0,p;p,0}(\Lambda_2)\, Q_{0,p;p,0}(\Lambda_1) \big) \right.$$

$$+\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda_2}{\Lambda_2}\int_{\Lambda_2}^{\Lambda_0}\frac{d\Lambda_1}{\Lambda_1}\sum_p Q_{0,p;p,0}(\Lambda_2)\,Q_{0,p;p,0}(\Lambda_1)\Bigg\} + \{\Lambda_1\leftrightarrow\Lambda_2\} + A_{00;00;00;00;00}(\Lambda_R)\;. \tag{2.11}$$

Thus, this definition necessitates a single initial value $A_{00;00;00;00;00}(\Lambda_R)$ which represents the normalization condition for the coupling constant.

Of particular importance are the marginal two-leg graphs with an accumulated index jump by 2, such as

$$\sum_{p,p',q,q'} \tag{2.12}$$



The corresponding initial value represents the normalization condition for the frequency parameter Ω in the initial action (1.2). Therefore, the harmonic oscillator potential must be present from the beginning in order to obtain a renormalizable model.

3 Bounds for the propagator

3.1 Propagator in the matrix base and cut-offs

The propagator of the noncommutative ϕ^4-model in the matrix base of the D-dimensional Moyal plane is given by[a] a positive sum [5], analogous to the heat-kernel or parametric α-space representation $\frac{1}{p^2+m^2} = \int_0^\infty d\alpha\, e^{-\alpha(p^2+m^2)}$ of the ordinary commutative propagator:

$$G_{m,m+h;l+h,l} = \frac{\theta}{8\Omega}\int_0^1 d\alpha\,\frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}+(\frac{D}{4}-1)}}{(1+C\alpha)^{\frac{D}{2}}}\prod_{s=1}^{\frac{D}{2}} G^{(\alpha)}_{m^s,m^s+h^s;l^s+h^s,l^s}\;, \tag{3.1}$$

$$G^{(\alpha)}_{m,m+h;l+h,l} = \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m+l+h}\sum_{u=\max(0,-h)}^{\min(m,l)}\mathcal{A}(m,l,h,u)\;\left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{m+l-2u}\;, \tag{3.2}$$

where $\mathcal{A}(m,l,h,u) = \sqrt{\binom{m}{m-u}\binom{m+h}{m-u}\binom{l}{l-u}\binom{l+h}{l-u}}$ and C is a function of Ω, namely $C(\Omega) = \frac{(1-\Omega)^2}{4\Omega}$. Indices such as m, l, h and u have $\frac{D}{2}$ non-negative components m^s, l^s, h^s, u^s, one for each symplectic pair of $\mathbb{R}^D$. However, due to (3.1) it is enough to prove estimations for a single component. We define the norm of an index by $\|m\| = \sum_{s=1}^{D/2} m^s$.

We know that cut-offs in the parametric representation for *commutative* theories are specially convenient both for perturbative and constructive renormalization. In the same

[a] Our representation (3.1) and (3.2) corresponds to (A.17) in [5] with $z = 1-\alpha$. The often used index parameter α in [5] is denoted by h.

spirit we will divide the integral (3.1) into slices. First we divide it into two different regions

- $M^{-1} \leqslant \alpha \leqslant 1$ where we expect an exponential decay in $m+l+h$ of order $\mathcal{O}(1)$,
- $0 \leqslant \alpha \leqslant M^{-1}$. This is the UV/IR region which is further sliced according to a geometric progression. For each slice we expect a scaled exponential decay.

The real number $M > 1$ has a carefully chosen Ω-dependence. Then, the decomposition

$$\int_0^1 d\alpha = \sum_{i=1}^{\infty} \int_{M^{-i}}^{M^{-i+1}} d\alpha \tag{3.3}$$

leads to the following propagator for the i^{th} slice:

$$G^i_{m,m+h,l+h,l} = \frac{\theta}{8\Omega} \int_{M^{-i}}^{M^{-i+1}} d\alpha \, \frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}+(\frac{D}{4}-1)}}{(1+C\alpha)^{\frac{D}{2}}} \prod_{s=1}^{\frac{D}{2}} G^{(\alpha)}_{m^s,m^s+h^s;l^s+h^s,l^s} \,. \tag{3.4}$$

The first slice $i=1$ is treated separately.

Remark that the factor $\mathcal{A}$ in (3.2) is the only one which prevents us from explicitly performing the u-sum. All the bounds in this paper are obtained by applying to the binomial coefficients in $\mathcal{A}$ the simple overestimate $\binom{n}{q} \leqslant \frac{n^q}{q!}$. Of course, this bound is sharp only for $q \ll n$. In the regime $n-q \ll n$ one should rather use the symmetric bound $\binom{n}{q} \leqslant \frac{n^{n-q}}{(n-q)!}$.

For $\alpha = 0$ we see from (3.2) that the propagator vanishes unless $u = l = m$. This suggests to bound $\mathcal{A}$ by

$$\mathcal{A}(m,l,h,u) \leqslant \frac{\sqrt{m(h+m)}^{m-u}\sqrt{l(h+l)}^{l-u}}{(m-u)!(l-u)!} \leqslant \frac{(m+h/2)^{m-u}(l+h/2)^{l-u}}{(m-u)!(l-u)!} \,. \tag{3.5}$$

Hence, for $\alpha \leqslant M^{-1}$,

$$G^{(\alpha)}_{m,m+h;l+h,l} \leqslant \left(\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+l+h} \times \sum_{u=\max(0,-h)}^{\min(m,l)} \frac{\left(\frac{\alpha(1-\Omega^2)\sqrt{m(m+h)}}{4\Omega\sqrt{1-\alpha}}\right)^{m-u}}{(m-u)!} \frac{\left(\frac{\alpha(1-\Omega^2)\sqrt{l(l+h)}}{4\Omega\sqrt{1-\alpha}}\right)^{l-u}}{(l-u)!} \,. \tag{3.6}$$

On the other hand, we observe from (3.2) that for $\alpha = 1$ the propagator vanishes unless $u = h = 0$. In this situation the bound (3.5) is not suitable anymore because $m-u$ and $l-u$ are of order $\mathcal{O}(m)$ and $\mathcal{O}(l)$, respectively. Instead, we can use

$$\mathcal{A}(m,l,h,u) \leqslant \frac{\sqrt{ml}^{u}\sqrt{(m+h)(l+h)}^{h+u}}{u!(h+u)!} \leqslant \frac{((m+l)/2)^u((m+l+2h)/2)^{h+u}}{u!(h+u)!} \,. \tag{3.7}$$

Inserting (3.7) into (3.2) we obtain

$$G^{(\alpha)}_{m,m+h;l+h,l} \leqslant \left(\frac{\alpha(1-\Omega^2)}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+l+h} \\ \times \sum_{u=\max(0,-h)}^{\min(m,l)} \frac{\left(\frac{4\Omega\sqrt{1-\alpha}\sqrt{ml}}{\alpha(1-\Omega^2)}\right)^u}{u!} \frac{\left(\frac{4\Omega\sqrt{1-\alpha}\sqrt{(m+h)(l+h)}}{\alpha(1-\Omega^2)}\right)^{u+h}}{(u+h)!} . \tag{3.8}$$

The further procedure will be to use the estimation

$$\sum_{u=0}^{\min(m,l)} \frac{X^{m-u}}{(m-u)!}\frac{Y^{l-u}}{(l-u)!} \leqslant e^{X+Y} . \tag{3.9}$$

Then, we have to find conditions on Ω and α under which a certain exponent is negative. These conditions are given in the following Lemma:

Lemma 1 *Let*

$$R(\Omega) := 1 - \frac{9}{10}\left(\left(\frac{1-\Omega}{1+\Omega}\right)\ln\left(\frac{1-\Omega}{1+\Omega}\right)\right)^2 \tag{3.10}$$

and Ω_R be the position of the maximum of $R(\Omega)$, i.e. $R'(\Omega_R)=0$. One has approximately $\Omega_R = 0.462117$. We define

$$M^{-1} = \begin{cases} R(\Omega) & \text{for } \Omega \geqslant \Omega_R \,, \\ R(\Omega_R) & \text{for } \Omega \leqslant \Omega_R \,. \end{cases} \tag{3.11}$$

Then, for all $\alpha \in [M^{-1},1]$ and all $\Omega \in (0,1)$ one has

$$E(\Omega,\alpha) := \frac{4\Omega\sqrt{1-\alpha}}{\alpha(1-\Omega^2)} + \ln\left(\frac{\alpha(1-\Omega^2)}{4\Omega+(1-\Omega)^2\alpha}\right) \leqslant -\frac{1}{15}\Omega M^{-1} . \tag{3.12}$$

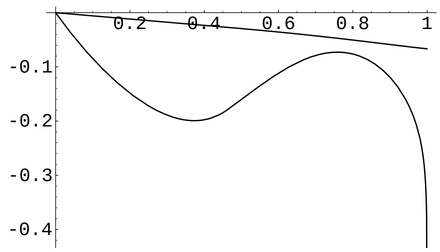
Proof. We have

$$\begin{aligned}\frac{\partial}{\partial\alpha}E(\Omega,\alpha) &= \frac{2\Omega}{(1-\Omega^2)\alpha^2\sqrt{1-\alpha}}\Big((\alpha-2) + \frac{2(1-\Omega^2)\alpha\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\Big) \\ &\leqslant \frac{2\Omega}{(1-\Omega^2)\alpha^2\sqrt{1-\alpha}}\Big((\alpha-2) + \frac{(1-\Omega^2)\alpha(2-\alpha)}{4\Omega+(1-\Omega)^2\alpha}\Big) \\ &= -\frac{2\Omega(2-\alpha)(2\Omega^2\alpha+(4-2\alpha)\Omega)}{(1-\Omega^2)\alpha^2\sqrt{1-\alpha}(4\Omega+(1-\Omega)^2\alpha)} < 0 .\end{aligned} \tag{3.13}$$

Thus, the function $E(\Omega,\alpha)$ is monotonously decreasing in α and, comparing $E(\Omega,1) = \ln\frac{1-\Omega}{1+\Omega} < 0$ with $E(\Omega,0)=+\infty$, has a single zero $E(\Omega,\alpha_0)=0$ with $\alpha_0 \approx 1$. Developing $E(\Omega,\alpha)$ about $\alpha=1$, the leading term is of order $\frac{1}{2}$:

$$E(\Omega,\alpha) := \sqrt{1-\alpha}\,\frac{4\Omega}{(1+\Omega)^2}\frac{(1+\Omega)}{(1-\Omega)} + \ln\frac{1-\Omega}{1+\Omega} + \mathcal{O}(1-\alpha) . \tag{3.14}$$

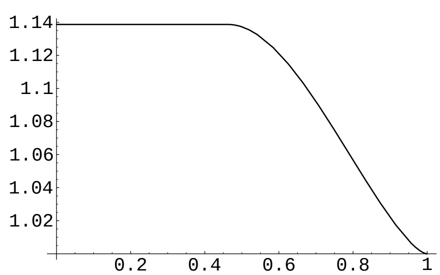
This means that for $\Omega \approx 1$, the zero is found near $\alpha_0 = 1 - \left(\frac{(1+\Omega)^2}{4\Omega} \frac{(1-\Omega)}{(1+\Omega)} \ln \frac{1-\Omega}{1+\Omega} \right)^2$. We know from (3.13) that $E(\Omega, \alpha) < 0$ for $\alpha \in (\alpha_0, 1]$. To be on the safe side with respect to higher order terms, we make the above estimation for α_0 slightly bigger by removing the factor $\frac{(1+\Omega)^2}{4\Omega} \geqslant 1$ and by rescaling the result by $\frac{9}{10}$. This leads to (3.10). We can now plot the function $E(\Omega, M^{-1})$ over Ω and compare it with $-\frac{1}{15}\Omega M^{-1}$:



Comparison of $E(\Omega, M^{-1})$ (the lower curve) with $-\frac{1}{15}\Omega M^{-1}$ (the upper curve), both plotted over Ω (3.15)

This finishes the proof. $\square$

For convenience we give a plot of the scale function M:



The scale function M plotted over Ω (3.16)

3.2 Main scaled bounds

The first result is to prove that the propagator shows a scaled exponential decay in any index. This is expressed by

Theorem 1 *There exists a constant K such that for $\Omega \in [0.5, 1)$, we have the uniform bound*

$$G^i_{m,m+h;l+h,l} \leqslant K M^{-i} e^{-\frac{\Omega}{15} M^{-i} \|m+l+h\|} , \tag{3.17}$$

where the scale parameter $M(\Omega) > 1$ is given by (3.11) and (3.10).

Proof. For the first slice $i = 1$ we use the bound (3.8). With (3.9) and $\sqrt{ml} \leqslant \frac{1}{2}(m+l)$ we obtain

$$G^{(\alpha)}_{m,m+h;l+h,l} \leqslant \exp\Big((m+l+h) E(\alpha, \Omega) \Big) . \tag{3.18}$$

Now, the bound (3.17) for the first slice $i = 1$ follows from Lemma 1, provided that M^{-1} is chosen according to (3.11) and (3.10).

Next, for $i \geqslant 2$, we use the bound (3.6), which with (3.9) can be brought into the form

$$G^{(\alpha)}_{m,m+h;l+h,l} \leqslant \exp\left(\alpha(m+l+h)\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right)\;. \tag{3.19}$$

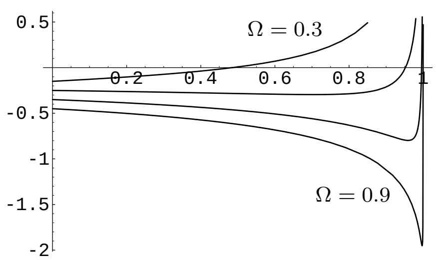
We have to prove that the exponent

$$\hat{E}_\beta(\Omega,\alpha) := \beta\frac{(1-\Omega^2)\alpha}{4\Omega\sqrt{1-\alpha}} + \ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha} \tag{3.20}$$

is negative for $\beta = 1$ and all $\alpha \leqslant M^{-1}$, where M^{-1} is determined by the first slice. One has $\hat{E}_\beta(\Omega, 0) = 0$ and

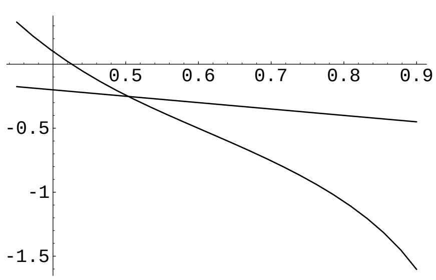
$$\frac{\partial}{\partial\alpha}\hat{E}_\beta(\Omega,\alpha) = \beta\frac{(1-\Omega^2)(2-\alpha)}{8\Omega\sqrt{(1-\alpha)^3}} - \frac{2(1+\Omega^2)-\alpha(1-\Omega)^2}{2(1-\alpha)(4\Omega+(1-\Omega)^2\alpha)}\;. \tag{3.21}$$

This implies $\frac{\partial}{\partial\alpha}\hat{E}_\beta(\Omega,\alpha)|_{\alpha=0} = \frac{(\beta-1)(1-\Omega^2)}{4\Omega} - \frac{\Omega}{2}$ and $\frac{\partial}{\partial\alpha}\hat{E}_\beta(\Omega,\alpha)|_{\alpha=1} = +\infty$. Thus, for Ω large enough and α small enough, we have $\frac{1}{\alpha}\hat{E}_\beta(\Omega,\alpha) < 0$. As α increases, $\hat{E}_\beta(\Omega,\alpha)$ will remain negative up to some α_β with $\hat{E}_\beta(\Omega,\alpha_\beta) = 0$. The next plot shows $\frac{1}{\alpha}\hat{E}_1(\Omega,\alpha)$ for certain values of Ω:



The function $\frac{1}{\alpha}E_1(\Omega,\alpha)$, for $\Omega \in \{0.3,\ 0.5,\ 0.7,\ 0.9\}$ plotted over α. The larger the value of Ω, the larger is the zero of $\frac{1}{\alpha}E_1(\Omega,\alpha)$. (3.22)

We have to ensure that $\alpha_1 > M^{-1}$. Thus, if $\hat{E}_\beta(\Omega, M^{-1}) < 0$, which requires an Ω large enough, there will exist a constant $c > 0$ such that $\frac{1}{\alpha}\hat{E}_\beta(\Omega,\alpha) \leqslant -c$ for all $\alpha \in [0, M^{-1}]$. The critical value for Ω is found when plotting $M\hat{E}_\beta(\Omega, M^{-1})$ over Ω. Comparing $M\hat{E}_1(\Omega, M^{-1})$ with the curve $-\frac{1}{2}\Omega$ relevant for $\alpha = 0$,



Comparison of $ME_1(\Omega, M^{-1})$ (the lower curve at large Ω) with $-\frac{1}{2}\Omega$, both plotted over Ω. (3.23)

we see that for $\Omega \geqslant 0.5$, the following estimation holds:

$$G^{(\alpha)}_{m,m+h,l+h,h} \leqslant e^{-\frac{1}{15}\Omega(m+l+h)\alpha}\;. \tag{3.24}$$

The Theorem now follows from (3.4), with $K = \frac{\theta(M_1-1)}{8\Omega}$. □

Theorem 2 *For the scale parameter M according to (3.11) and (3.10) there exists a constant*[b] *K such that for all $\Omega \in [0.5, 1)$ we have the uniform bound*

$$
\begin{aligned}
&G^i_{m,m+h;l+h,l} \\
&\leqslant \frac{K}{M^i} e^{-\frac{\Omega}{15} M^{-i}\|m+l+h\|} \prod_{s=1}^{\frac{D}{2}} \min\left(1, \left(\frac{1+\min(m^s, l^s, m^s+h^s, l^s+h^s)}{M^i/5}\right)^{\frac{|m^s-l^s|}{2}}\right) . \qquad (3.25)
\end{aligned}
$$

Proof. Of course, this bound improves (3.17) only when an index component is smaller than $M^i/5$. We can, therefore, assume that $i > 12$. In particular, there is nothing to prove for the first slice $i = 1$.

Suppose $l \leqslant m \leqslant m+h$ and $\delta = m-l$. Instead of (3.9) we use the improved estimation

$$
\begin{aligned}
\sum_{u=0}^{l} \frac{X^{m-u}}{(m-u)!} \frac{Y^{l-u}}{(l-u)!} &= \sum_{v=0}^{l} \frac{X^{m-l+v}}{(m-l+v)!} \frac{Y^v}{v!} \\
&\leqslant \frac{X^{m-l}}{(m-l)!} \sum_{v=0}^{l} \left(\frac{1}{v!}\right)^2 (XY)^v \leqslant \frac{X^{m-l}}{(m-l)!} e^{X+Y} . \qquad (3.26)
\end{aligned}
$$

Then, the propagator (3.6) takes the form

$$
\begin{aligned}
G^{(\alpha)}_{m,m+h;l+h,l} &\leqslant \exp\left((m+l+h)\alpha\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right) \\
&\times \sqrt{\frac{\left(\frac{m\alpha(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}(2\beta-2)}\right)^{\delta}}{\delta!}} \sqrt{\frac{\left(\frac{(2\beta-2)(m+h)\alpha(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}}\right)^{\delta}}{\delta!}} . \qquad (3.27)
\end{aligned}
$$

We choose $\beta = \frac{5}{4}$, but any choice $\beta > 1$ would be possible. In the last term (containing $m+h$) we estimate $\frac{x^\delta}{\delta!} \leqslant e^x$ and add the exponential to the first line of (3.27). In the first term of the last line of (3.27) we use the estimation

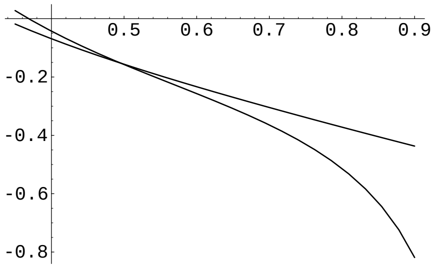
$$
\frac{1}{\delta!} \leqslant \left(\frac{\delta}{e}\right)^{-\delta} . \qquad (3.28)
$$

This yields

$$
\begin{aligned}
&G^{(\alpha)}_{m,m+h;l+h,l} \\
&\leqslant \left(\frac{m\alpha(1-\Omega^2)e}{2\Omega\delta\sqrt{1-\alpha}}\right)^{\frac{\delta}{2}} \exp\left((m+l+h)\alpha\left(\frac{5}{4}\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right) . \qquad (3.29)
\end{aligned}
$$

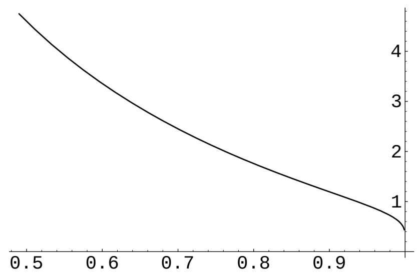
[b]In the following, the K's will be kinds of "dustbin" constants. It means that their contents changes whereas their names do not.

For $\beta = \frac{5}{4}$ the exponent in (3.29) is negative for $\Omega \geqslant 0.5$:



Comparison of $M^{12}E_{\frac{5}{4}}(\Omega, M^{-12})$ (the lower curve at large Ω) with $\frac{1-\Omega^2}{16\Omega} - \frac{1}{2}\Omega$, both plotted over Ω. (3.30)

Next, for $i \geqslant 12$ and $\Omega \geqslant 0.5$, one has $\frac{(1-\Omega^2)e}{\Omega\sqrt{1-M^{-i}}} \leqslant 5$, as the following plot shows:



The function $\frac{(1-\Omega^2)e}{\Omega\sqrt{1-M^{-12}}}$ plotted over Ω. (3.31)

Now we are left with two cases:

a) $l = 0 \Leftrightarrow m = \delta$:

$$G^{(\alpha)}_{m,m+h;l+h,l} \leqslant e^{-\frac{\Omega}{15}(m+l+h)\alpha} \left(\tfrac{5}{2}\alpha\right)^{\delta/2} . \tag{3.32}$$

b) $l \geqslant 1$ and $\delta \geqslant 1$: using $l + \delta \leqslant 2l\delta$,

$$G^{(\alpha)}_{m,m+h;l+h,l} \leqslant e^{-\frac{\Omega}{15}(m+l+h)\alpha} (5l\alpha)^{\delta/2} . \tag{3.33}$$

Inserting this into (3.4) and symmetrizing with respect to the smallest index we obtain (3.25) with $K = \frac{\theta(M-1)}{8\Omega}$. $\square$

Let us now consider a typical graph appearing in the process of renormalization, that is, with external legs carrying indices *lower* than the internal ones. The bound (3.25) provides a good factor with respect to power-counting unless the index jump $\delta = |m - l|$ is very small, typically $\delta = 0$ or $\delta = 1$. This ensures that if the lower index of a propagator is smaller than the scale we look at, the index is conserved along its trajectory for power-counting relevant and marginal graphs.

Unfortunately, that estimation does not carry any information when the lower index is larger than the scale. It leads to a difficulty for graphs which possess completely inner vertices. Therefore, we have to find estimates for propagators with a sum over the index l. The next section is devoted to these bounds.

3.3 Bounds for sums

Now we want to prove that the summation of the propagator $G^{(\alpha)}_{m,p-l,p,m+l}$ over l, for m and p kept constant, gives the same power-counting as in the previous section. The proof relies on a more accurate estimate of the sum in (3.9).

Theorem 3 *For $\Omega \in [0.5, 1)$ there exists a constant K such that we have the uniform bound*

$$\sum_{l=-m}^{p} G^{i}_{m,p-l,p,m+l} \leqslant K M^{-i}\, e^{-\frac{\Omega}{20}M^{-i}(\|p\|+\|m\|)} \; . \tag{3.34}$$

Proof. The first slices, say $i \leqslant 16$, are trivial to treat. Using (3.17) we have

$$\sum_{l=-m}^{p} G^{i}_{m,p-l,p,m+l} \leqslant \frac{K}{M^{i}} \prod_{s=1}^{2} (m^{s}+p^{s}+1) e^{-\frac{\Omega M^{-i}}{15}(m^{s}+p^{s})} \; . \tag{3.35}$$

Then, the estimation follows from

$$(x+1)e^{-\frac{1}{15}\Omega M^{-i}x} \leqslant \left(\frac{60M^{i}}{\Omega} e^{\frac{\Omega M^{-i}}{60}-1}\right) e^{-\frac{1}{20}\Omega M^{-i}x} \; . \tag{3.36}$$

This method fails in the limit $i \to \infty$. Thus, for large i, we have to estimate the propagator (3.6) more carefully, now putting $h \mapsto p-m-l$ and $l \mapsto m+l$. Without loss of generality we can assume $p \geqslant m$. We have to divide the range of summation into three parts according to the smallest index. Using (3.26) we estimate

$$\begin{aligned}
&\sum_{l=-m}^{p} G^{(\alpha)}_{m,p-l;p,m+l} \\
&\leqslant \left(\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+p} \left(\sum_{l=-m}^{-1} \sum_{u=0}^{m+l} \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+l-u}}{(m+l-u)!} \right. \\
&\left. + \sum_{l=0}^{p-m-1} \sum_{u=0}^{m} \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+l-u}}{(m+l-u)!} + \sum_{l=p-m}^{p} \sum_{v=0}^{p-l} \frac{X^{p-l-v}}{(p-l-v)!} \frac{Y^{p-v}}{(p-v)!} \right) \\
&\leqslant \left(\sum_{l=-m}^{-1} \frac{X^{|l|}}{|l|!} + \sum_{l=0}^{p} \frac{Y^{l}}{l!} \right) \exp\left((m+p)\left(\frac{\alpha(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \ln \frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha} \right)\right) \\
&\leqslant 2 \underbrace{\sum_{l=0}^{p} \frac{\left(\frac{\alpha(1-\Omega^2)(m+p+l)}{8\Omega\sqrt{1-\alpha}}\right)^{l}}{l!}}_{Z} \exp\left((m+p)\alpha \left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha} \ln \frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha} \right)\right) \; ,
\end{aligned} \tag{3.37}$$

where $X = \frac{\alpha(1-\Omega^2)(p+m-l)}{8\Omega\sqrt{1-\alpha}}$ and $Y = \frac{\alpha(1-\Omega^2)(p+m+l)}{8\Omega\sqrt{1-\alpha}}$.

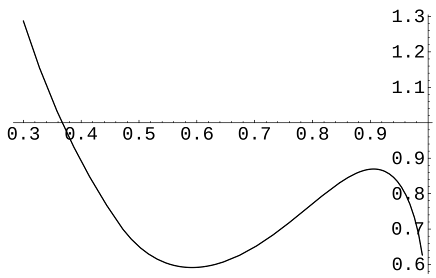
We can now divide the sum over l into two regions corresponding to $l \leqslant (2\beta-3)(p+m)$ and $l \geqslant \lceil(2\beta-3)(p+m)\rceil \geqslant (2\beta-3)(p+m)$, where $\lceil x\rceil$ is the smallest integer which is larger than x and $\beta > \frac{3}{2}$ will be determined later:

$$Z \leqslant \sum_{l=0}^{(2\beta-3)(p+m)} \frac{\left(\frac{\alpha(1-\Omega^2)(\beta-1)(m+p)}{4\Omega\sqrt{1-\alpha}}\right)^l}{l!} + \sum_{l=\lceil(2\beta-3)(p+m)\rceil}^{p} \frac{\left(\frac{\alpha(1-\Omega^2)(\beta-1)l}{4\Omega\sqrt{1-\alpha}(2\beta-3)}\right)^l}{l!} \,. \tag{3.38}$$

We extend both sums to infinity and use in the second one the identity (3.28):

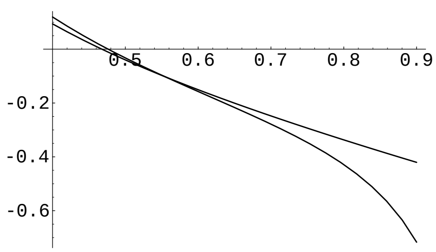
$$Z \leqslant \exp\left(\frac{\alpha(1-\Omega^2)(\beta-1)(m+p)}{4\Omega\sqrt{1-\alpha}}\right) + \sum_{l=0}^{\infty}\left(\frac{\alpha(1-\Omega^2)e(\beta-1)}{4\Omega\sqrt{1-\alpha}(2\beta-3)}\right)^l \,. \tag{3.39}$$

For $\beta = \frac{39}{25}$ one confirms $\frac{M^{-16}(1-\Omega^2)e(\beta-1)}{4\Omega\sqrt{1-M^{-16}}(2\beta-3)} < 1$ for $\Omega \geqslant 0.5$:



The function $\frac{M^{-16}(1-\Omega^2)e(\beta-1)}{4\Omega\sqrt{1-M^{-16}}(2\beta-3)}$ for $\beta = \frac{39}{25}$ plotted over Ω. (3.40)

On the other hand, the following plot shows that $\frac{39}{25}\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha} < -\frac{1}{20}\Omega$ for $\alpha \leqslant M^{-16}$ and $\Omega \geqslant 0.5$:



Comparison of $M^{16}E_{\frac{39}{25}}(\Omega, M^{-16})$ (the lower curve at large Ω) with $\frac{7(1-\Omega^2)}{50\Omega} - \frac{1}{2}\Omega$, both plotted over Ω. (3.41)

This finishes the proof. $\square$

The previous estimation for the summed propagator is still not enough for the renormalization proof, because the index sums are entangled in the graph. We have to prove that the exponential decay is still achieved if for given summation variable l we maximise the other index p:

Theorem 4 *For $\Omega \geqslant 0.58$ there exists a constant K such that we have the uniform bound*

$$\sum_{l=-m}^{\infty} \max_{p\geqslant\max(l,0)} G^i_{m,p-l;p,m+l} \leqslant KM^{-i}e^{-\frac{\Omega}{36}M^{-i}\|m\|} \,. \tag{3.42}$$

Proof. Again, the main estimate (3.17) guarantees the desired behavior (3.42) for the first slices, say $i \leqslant 16$. For $l \leqslant 0$ the maximum is attained at $p = 0$ so that we are in the situation of (3.35) and (3.36). For $l > 0$ the maximum is attained at $p = l$ so that the l-sum leads to a geometric series. Here, it is important that i is bounded.

For $i > 16$ we have to proceed differently. We divide the domain of summation according to the smallest index at the propagator:

$$\sum_{l=-m}^{\infty} \max_{p\geqslant \max(l,0)} = \sum_{l=-m}^{-1} \max_{0\leqslant p<m+l} + \sum_{l=-m}^{-1} \max_{m+l\leqslant p<m} + \sum_{l=-m}^{-1} \max_{p\geqslant m} + \sum_{l=0}^{\infty} \max_{l\leqslant p<m+l} + \sum_{l=0}^{\infty} \max_{p\geqslant m+l} . \tag{3.43}$$

We now obtain from (3.6) and (3.26)

$$\begin{aligned}
&\sum_{l=-m}^{\infty} \max_{p\geqslant \max(l,0)} G^{(\alpha)}_{m,p-l;p,m+l} \\
&\leqslant \sum_{l=-m}^{-1} \max_{0\leqslant p<m+l} \left(\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+p} \sum_{v=0}^{p} \frac{X^{p-l-v}}{(p-l-v)!}\frac{Y^{p-v}}{(p-v)!} \\
&+ \sum_{l=-m}^{-1} \max_{m+l\leqslant p<m} \left(\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+p} \sum_{u=0}^{m+l} \frac{X^{m-u}}{(m-u)!}\frac{Y^{m+l-u}}{(m+l-u)!} \\
&+ \sum_{l=-m}^{-1} \max_{p\geqslant m} \left(\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+p} \sum_{u=0}^{m+l} \frac{X^{m-u}}{(m-u)!}\frac{Y^{m+l-u}}{(m+l-u)!} \\
&+ \sum_{l=0}^{\infty} \max_{l\leqslant p<m+l} \left(\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+p} \sum_{v=0}^{p-l} \frac{X^{p-l-v}}{(p-l-v)!}\frac{Y^{p-v}}{(p-v)!} \\
&+ \sum_{l=0}^{\infty} \max_{p\geqslant m+l} \left(\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)^{m+p} \sum_{u=0}^{m} \frac{X^{m-u}}{(m-u)!}\frac{Y^{m+l-u}}{(m+l-u)!} \\
&\leqslant \sum_{|l|=1}^{m} \max_{p\geqslant 0} \exp\left((p+m)\alpha\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right) \frac{\left(\frac{\alpha(1-\Omega^2)(p+m+|l|)}{8\Omega\sqrt{1-\alpha}}\right)^{|l|}}{|l|!}
\end{aligned} \tag{3.44}$$

$$+ \sum_{l=0}^{\infty} \max_{p\geqslant l} \exp\left((p+m)\alpha\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right) \frac{\left(\frac{\alpha(1-\Omega^2)(p+m+l)}{8\Omega\sqrt{1-\alpha}}\right)^{l}}{l!} , \tag{3.45}$$

where $X = \frac{\alpha(1-\Omega^2)(p+m-l)}{8\Omega\sqrt{1-\alpha}}$ and $Y = \frac{\alpha(1-\Omega^2)(p+m+l)}{8\Omega\sqrt{1-\alpha}}$.

The function $e^{-\gamma p}\frac{(p+q)^l}{l!}$ attains its maximum $e^{\gamma q-l}\frac{1}{l!}\left(\frac{l}{\gamma}\right)^l$ at $p=\frac{l}{\gamma}-q$. We need this property for $q=m+|l|$ and

$$\gamma := -\frac{\alpha(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} - \ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha} > 0\ . \tag{3.46}$$

However, we have to take into account the range of p. If $\frac{l}{\gamma}-q<0$ in (3.44), then the function $e^{-\gamma p}\frac{(p+q)^l}{l!}$ is monotonously decreasing for all $p\geqslant 0$ so that the maximum is at $p=0$. Otherwise we have to insert the specific maximum. This means that we split the sum over l in (3.44) into two pieces

- $|l|\leqslant\frac{m\gamma}{1-\gamma}$, where we insert $p=0$. Actually, we can safely extend this sum from 0 to m, still keeping $p=0$.
- $|l|\geqslant\frac{m\gamma}{1-\gamma}$, where we insert $p=\frac{1-\gamma}{\gamma}|l|-m$.

This gives

$$\begin{aligned}
&\sum_{|l|=1}^{m}\max_{p\geqslant 0}\exp\left((p+m)\alpha\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}}+\frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right)\frac{\left(\frac{\alpha(1-\Omega^2)(p+m+|l|)}{8\Omega\sqrt{1-\alpha}}\right)^{|l|}}{|l|!}\\
&\leqslant\sum_{l=0}^{m}\exp\left(m\alpha\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}}+\frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right)\frac{\left(\frac{\alpha(1-\Omega^2)(m+l)}{8\Omega\sqrt{1-\alpha}}\right)^{l}}{l!}\\
&+\sum_{l=\frac{\gamma m}{1-\gamma}}^{m}\exp\left(-l-\alpha l\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}}+\frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right)\frac{\left(\frac{\alpha(1-\Omega^2)l}{8\Omega\gamma\sqrt{1-\alpha}}\right)^{l}}{l!}\ .
\end{aligned}\tag{3.47}$$

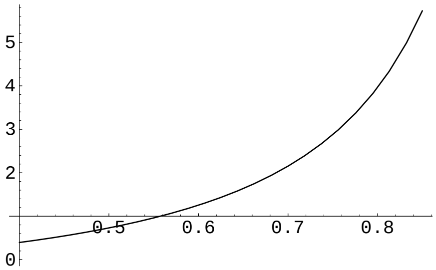
For the first sum we are in the situation of (3.37) with $m\mapsto 0, p\mapsto m$ so that we can bound that sum according to the steps leading to (3.34) by a constant times $e^{-\frac{1}{20}\Omega m\alpha}$, for $\Omega\geqslant 0.5$ and $\alpha\leqslant M^{-16}$. In the second sum we use (3.28) so that, for some number $\varepsilon>0$,

$$\begin{aligned}
&\sum_{l=\frac{\gamma m}{1-\gamma}}^{m}\exp\left(-l-\alpha l\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}}+\frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right)\frac{\left(\frac{\alpha(1-\Omega^2)l)}{8\Omega\gamma\sqrt{1-\alpha}}\right)^{l}}{l!}\\
&\leqslant\sum_{l=\frac{\gamma m}{1-\gamma}}^{m}e^{-\varepsilon l}\left(\frac{\alpha(1-\Omega^2)e^{\varepsilon+\gamma}}{8\Omega\gamma\sqrt{1-\alpha}}\right)^l\leqslant e^{-\varepsilon\gamma m}\sum_{l=\frac{\gamma m}{1-\gamma}}^{m}\left(\frac{\alpha(1-\Omega^2)e^{\varepsilon+\gamma}}{8\Omega\gamma\sqrt{1-\alpha}}\right)^l\ .
\end{aligned}\tag{3.48}$$

In the numerator we can bound γ by $\frac{1}{2}$, otherwise the sum vanishes. Moreover, we choose $\varepsilon=\frac{1}{16}$. We then see from (3.46) that the following condition is to prove:

$$\frac{8\Omega\gamma\sqrt{1-\alpha}}{\alpha(1-\Omega^2)e^{\frac{9}{16}}}=\frac{8\Omega\sqrt{1-\alpha}}{\alpha(1-\Omega^2)e^{\frac{9}{16}}}\ln\frac{4\Omega+(1-\Omega)^2\alpha}{4\Omega\sqrt{1-\alpha}}-\frac{2}{e^{\frac{9}{16}}}>1\ . \tag{3.49}$$

This function is monotonously decreasing in α so that it is sufficient to check it for $\alpha = M^{-16}$:



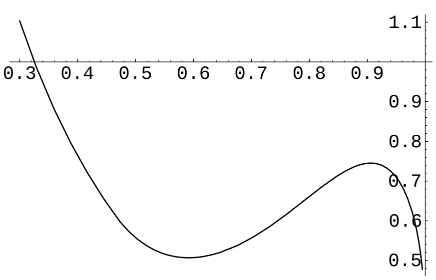
The function $\frac{8\Omega\sqrt{1-M^{-16}}}{M^{-16}(1-\Omega^2)e^{\frac{9}{16}}} \ln \frac{4\Omega+(1-\Omega)^2 M^{-16}}{4\Omega\sqrt{1-M^{-16}}} - \frac{2}{e^{\frac{9}{16}}}$ plotted over Ω. (3.50)

We see that it is sufficient to have $\Omega \geqslant 0.58$. Moreover, $-\varepsilon\gamma m = \frac{\alpha}{16} m(\frac{1}{\alpha}\hat{E}_1(\Omega,\alpha))$, where $\hat{E}_1$ is given in (3.20). We then know from (3.23) that for $\alpha \in [0, M^{-16}]$ and $\Omega \geqslant 0.58$, $-\varepsilon\gamma m \leqslant -\frac{1}{32}\alpha m$. This shows that (3.44) leads to the desired estimation (3.42).

We now pass to (3.45). The condition $p \geqslant l$ leads to a splitting of the l-sum at $\frac{\gamma m}{1-2\gamma}$. For smaller l we have $p = l$:

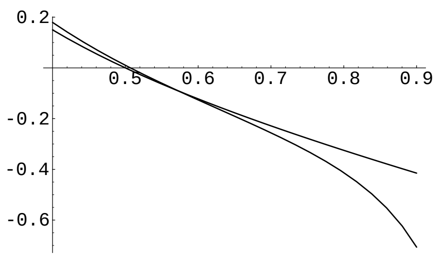
$$
\begin{aligned}
&\sum_{l=0}^{\infty} \max_{p \geqslant l} \exp\left((p+m)\alpha\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right) \frac{\left(\frac{\alpha(1-\Omega^2)(p+m+|l|)}{8\Omega\sqrt{1-\alpha}}\right)^l}{l!} \\
&\leqslant \sum_{l=0}^{\infty} \exp\left((m+l)\alpha\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right) \frac{\left(\frac{\alpha(1-\Omega^2)(m+2l)}{8\Omega\sqrt{1-\alpha}}\right)^l}{l!} \\
&+ \sum_{l=\frac{\gamma m}{1-2\gamma}}^{\infty} \exp\left(-l-\alpha l\left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}\right)\right) \frac{\left(\frac{\alpha(1-\Omega^2)l}{8\Omega\gamma\sqrt{1-\alpha}}\right)^l}{l!}\;.
\end{aligned} \tag{3.51}
$$

The second sum on the right hand side is identical to treat as in the previous case (3.44). The first sum on this right hand side is split as in (3.38) at $l = (\beta - \frac{3}{2})m$. Compared with (3.39) we now have to achieve $\frac{M^{-16}(1-\Omega^2)e(2\beta-2)}{4\Omega\sqrt{1-M^{-16}}(2\beta-3)} < 1$ for $\Omega \geqslant 0.58$. We can take $\beta = \frac{5}{3}$:



The function $\frac{M^{-16}(1-\Omega^2)e(\beta-1)}{4\Omega\sqrt{1-M^{-16}}(2\beta-3)}$ for $\beta = \frac{5}{3}$ plotted over Ω. (3.52)

On the other hand, the following plot shows that $\frac{5}{3}\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}}+\frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+(1-\Omega)^2\alpha}<-\frac{1}{36}\Omega$ for $\alpha\leqslant M^{-16}$ and $\Omega\geqslant 0.58$:



Comparison of $M^{16}E_{\frac{5}{3}}(\Omega,M^{-16})$ (the lower curve at large Ω) with $\frac{(1-\Omega^2)}{6\Omega}-\frac{1}{2}\Omega$, both plotted over Ω. (3.53)

The l-dependence of the argument of the exponential in the second line of (3.51) can be ignored. This finishes the proof. $\square$

3.4 Composite propagators

This section is devoted to the proofs of bounds on the *composite propagators* introduced in [5]. We define their sliced versions as follows:

$$\mathcal{Q}^{i(0)}_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}=G^i_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}-G^i_{\substack{0\\0},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{0\\0}}\;, \tag{3.54}$$

$$\mathcal{Q}^{i(1)}_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}=\mathcal{Q}^{i(0)}_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}-m^1\mathcal{Q}^{i(0)}_{\substack{1\\0},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{1\\0}}-m^2\mathcal{Q}^{i(0)}_{\substack{0\\1},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{0\\1}}\;, \tag{3.55}$$

$$\mathcal{Q}^{i(\frac{1}{2})}_{\substack{m^1+1\\m^2},\substack{n^1+1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}=G^i_{\substack{m^1+1\\m^2},\substack{n^1+1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}-\sqrt{m^1+1}\,G^i_{\substack{1\\0},\substack{n^1+1\\n^2};\substack{n^1\\n^2},\substack{0\\0}}\;. \tag{3.56}$$

Theorem 5 *For M according to (3.11) and (3.10) there exist constants K_i such that for $\Omega\in[0.5,1)$ and $m\leqslant M^i$, we have the uniform bounds*

$$\Big|\mathcal{Q}^{i(0)}_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}\Big|\leqslant K_0M^{-i}\,\frac{m^1+m^2}{M^i}\,e^{-\frac{\Omega}{15}M^{-i}(n^1+n^2)}\;, \tag{3.57}$$

$$\Big|\mathcal{Q}^{i(1)}_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}\Big|\leqslant K_1M^{-i}\,\Big(\frac{m^1+m^2}{M^i}\Big)^2\,e^{-\frac{\Omega}{15}M^{-i}(n^1+n^2)}\;, \tag{3.58}$$

$$\Big|\mathcal{Q}^{i(\frac{1}{2})}_{\substack{m^1+1\\m^2},\substack{n^1+1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}\Big|\leqslant K_2M^{-i}\,\Big(\frac{m^1+m^2+1}{M^i}\Big)^{3/2}\,e^{-\frac{\Omega}{15}M^{-i}(n^1+n^2)}\;. \tag{3.59}$$

Proof. There is no need to discuss the first slice. From (3.4) we have

$$\begin{aligned}\mathcal{Q}^{i(0)}_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}=\frac{\theta}{8\Omega}\int_{M^{-i}}^{M^{-i+1}}d\alpha\,\frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}}}{(1+C\alpha)^2}\Big(&\big(G^{(\alpha)}_{m^1,n^1,n^1,m^1}-G^{(\alpha)}_{0,n^1,n^1,0}\big)G^{(\alpha)}_{m^2,n^2,n^2,m^2}\\&+G^{(\alpha)}_{0,n^1,n^1,0}\big(G^{(\alpha)}_{m^2,n^2,n^2,m^2}-G^{(\alpha)}_{0,n^2,n^2,0}\big)\Big)\;.\end{aligned} \tag{3.60}$$

Taking (3.17) into account, it remains to obtain estimations for $G^{(\alpha)}_{m,n,n,m} - G^{(\alpha)}_{0,n,n,0}$. From (3.2) we have, expressing $m+h$ by n,

$$\begin{aligned}
&\big|G^{(\alpha)}_{m,n,n,m} - G^{(\alpha)}_{0,n,n,0}\big| \\
&= \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{n} \left| \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m} \sum_{u=0}^{\min(m,n)} \binom{m}{u}\binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{2u} - 1 \right| . \qquad (3.61)
\end{aligned}$$

The $(u=0)$-part of the sum and the separate term -1 yield

$$\begin{aligned}
1 - \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m} &= \left(1 - \frac{\sqrt{1-\alpha}}{1+C\alpha}\right) \sum_{j=0}^{m-1} \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{j} \\
&\leqslant m \left(1 - \frac{\sqrt{1-\alpha}}{1+C\alpha}\right) \\
&\leqslant m\alpha(C+1) \,. \qquad (3.62)
\end{aligned}$$

Next, using $\binom{m}{v+1} = \frac{m-v}{v+1}\binom{m}{v}$ and our central estimate $\binom{m}{v} \leqslant \frac{m^v}{v!}$, we have

$$\begin{aligned}
&\sum_{u=1}^{m} \binom{m}{u}\binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{2u} \\
&\leqslant \sum_{v=0}^{m-1} \frac{m-v}{v+1}\binom{m}{v} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{v+1} \sum_{u=0}^{m} \binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{u} \\
&\leqslant m \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right) \exp\left(\frac{C\alpha(1+\Omega)(m+n)}{\sqrt{1-\alpha}\,(1-\Omega)}\right) \\
&\leqslant m\alpha \frac{(1-\Omega)^2}{4\Omega\sqrt{1-M^{-1}}} \exp\left(\frac{\alpha(1-\Omega^2)(m+n)}{4\Omega\sqrt{1-\alpha}}\right) . \qquad (3.63)
\end{aligned}$$

Inserting (3.62) and (3.63) back into (3.61) we obtain for $\alpha \leqslant M^{-1}$ the estimation

$$\begin{aligned}
\big|G^{(\alpha)}_{m,n,n,m} - G^{(\alpha)}_{0,n,n,0}\big| &\leqslant \alpha m \left(\frac{(1+\Omega)^2}{4\Omega} + \frac{(1-\Omega)^2}{4\Omega\sqrt{1-M^{-1}}}\right) \\
&\quad \times \exp\left(\alpha n \left(\frac{(1-\Omega^2)}{4\Omega\sqrt{1-\alpha}} + \frac{1}{\alpha}\ln\frac{4\Omega\sqrt{1-\alpha}}{4\Omega+\alpha(1-\Omega)^2}\right)\right) . \qquad (3.64)
\end{aligned}$$

Comparing (3.64) with (3.19) we obtain in (3.57) the same restrictions to Ω as in Theorem 1.

To approach (3.58) we consider

$$\mathcal{Q}^{i(1)}_{\substack{m^1\\m^2},\substack{n^1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}} = \frac{\theta}{8\Omega}\int_{M^{-i}}^{M^{-i+1}} d\alpha\,\frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}}}{(1+C\alpha)^2}
\times\Big(\Big(G^{(\alpha)}_{m^1,n^1,n^1,m^1}-G^{(\alpha)}_{0,n^1,n^1,0}-m^1\big(G^{(\alpha)}_{1,n^1,n^1,1}-G^{(\alpha)}_{0,n^1,n^1,0}\big)\Big)G^{(\alpha)}_{m^2,n^2,n^2,m^2}
+m^1\big(G^{(\alpha)}_{1,n^1,n^1,1}-G^{(\alpha)}_{0,n^1,n^1,0}\big)\big(G^{(\alpha)}_{m^2,n^2,n^2,m^2}-G^{(\alpha)}_{0,n^2,n^2,0}\big)
+G^{(\alpha)}_{0,n^1,n^1,0}\Big(G^{(\alpha)}_{m^2,n^2,n^2,m^2}-G^{(\alpha)}_{0,n^2,n^2,0}-m^2\big(G^{(\alpha)}_{1,n^2,n^2,1}-G^{(\alpha)}_{0,n^2,n^2,0}\big)\Big)\Big)\,. \tag{3.65}$$

The third line is estimated by (3.64). In the other lines we have for $n \geqslant 1$

$$\begin{aligned}
&\big|G^{(\alpha)}_{m,n,n,m}-G^{(\alpha)}_{0,n,n,0}-m(G^{(\alpha)}_{1,n,n,1}-G^{(\alpha)}_{0,n,n,0})\big|\\
&=\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}\Big)^{n}\,\Bigg|\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}\Big)^{m}-m\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}-1\Big)-1\Bigg|\\
&+mn\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}\Big)^{n+1}\Big(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\Big)^{2}\,\Bigg|\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}\Big)^{m-1}-1\Bigg|\\
&+\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}\Big)^{m+n}\sum_{u=2}^{\min(m,n)}\binom{m}{u}\binom{n}{u}\Big(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\Big)^{2u}\,.
\end{aligned} \tag{3.66}$$

In the third line we use $n\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)} \leqslant \exp\Big(n\alpha\frac{(1-\Omega)^2}{4\Omega\sqrt{1-\alpha}}\Big)$. The further procedure is as before.

Finally, we consider

$$\begin{aligned}
&\mathcal{Q}^{i(\frac{1}{2})}_{\substack{m^1+1\\m^2},\substack{n^1+1\\n^2};\substack{n^1\\n^2},\substack{m^1\\m^2}}\\
&=\frac{\theta}{8\Omega}\int_{M^{-i}}^{M^{-i+1}} d\alpha\,\frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}}}{(1+C\alpha)^2}\Big(\big(G^{(\alpha)}_{m^1+1,n^1+1,n^1,m^1}-\sqrt{m^1+1}G^{(\alpha)}_{1,n^1+1,n^1,0}\big)G^{(\alpha)}_{m^2,n^2,n^2,m^2}\\
&\qquad+\sqrt{m^1+1}G^{(\alpha)}_{1,n^1+1,n^1,0}\big(G^{(\alpha)}_{m^2,n^2,n^2,m^2}-G^{(\alpha)}_{0,n^2,n^2,0}\big)\Big)\,.
\end{aligned} \tag{3.67}$$

The estimation for the second line of (3.67) is immediately obtained from (3.64) and (3.25). In the second line we have

$$\begin{aligned}
&\big|G^{(\alpha)}_{m+1,n+1,n,m}-\sqrt{m+1}G^{(\alpha)}_{1,n+1,n,0}\big|\\
&=\Bigg|\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}\Big)^{m}\sum_{u=0}^{\min(m,n)}\frac{\sqrt{(m+1)(n+1)}}{u+1}\binom{m}{u}\binom{n}{u}\Big(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\Big)^{2u+1}\\
&\qquad-\sqrt{(m+1)(n+1)}\Big(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\Big)\Bigg|\Big(\frac{\sqrt{1-\alpha}}{1+C\alpha}\Big)^{n+1}\,.
\end{aligned} \tag{3.68}$$

We use the estimation $\sqrt{n\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}} \leqslant \exp\left(\frac{1}{2}n\alpha\frac{(1-\Omega)^2}{4\Omega\sqrt{1-\alpha}}\right)$ and proceed along the same lines as before.

This finishes the proof. $\square$

4 Perturbative power-counting

4.1 Ribbon graphs

Consider a given ϕ^4-ribbon graph G with N external legs, V vertices, I inner lines and F faces, hence of genus $g = 1 - \frac{1}{2}(V - I + F)$. There are four indices $\{m, n; k, l\} \in \mathbb{N}^2$ associated to each inner line $\underset{m \quad l}{\overset{n \quad k}{\overleftarrow{\underrightarrow{\quad\quad}}}}$ of the graph and two indices for each external line, hence $4I + 2N = 8V$ such indices. But at every vertex, the left index of a ribbon coincides with the right one of the next ribbon. This creates $4V$ independent identifications, so we can write the indices of any propagator in terms of a set $\mathcal{I}$ of $4V$ indices, four per vertex, for instance each "left" half-ribbon index.

The amplitude of the graph is then the sum

$$A_G = \sum_{\mathcal{I}} \prod_{\delta \in G} G_{m_\delta(\mathcal{I}),n_\delta(\mathcal{I});k_\delta(\mathcal{I}),l_\delta(\mathcal{I})}\ \delta_{m_\delta - l_\delta, n_\delta - k_\delta}\ , \tag{4.1}$$

where the four basic indices of the propagator G for a line δ are functions of $\mathcal{I}$ called $\{m_\delta(\mathcal{I}), n_\delta(\mathcal{I}); k_\delta(\mathcal{I}), l_\delta(\mathcal{I})\}$.

The scale decomposition of the propagator being

$$G = \sum_{i=0}^{\infty} G^i\ , \tag{4.2}$$

we have an associated decomposition of any amplitude of the theory as

$$A_G = \sum_{\mu} A_{G,\mu}\ , \tag{4.3}$$

$$A_{G,\mu} = \sum_{\mathcal{I}} \prod_{\delta \in G} G^{i_\delta}_{m_\delta(\mathcal{I}),n_\delta(\mathcal{I});k_\delta(\mathcal{I}),l_\delta(\mathcal{I})}\ \delta_{m_\delta(\mathcal{I}) - l_\delta(\mathcal{I}), n_\delta(\mathcal{I}) - k_\delta(\mathcal{I})}\ , \tag{4.4}$$

where $\mu = \{i_\delta\}$ runs over all possible attributions of a positive integer i_δ for each line δ. Such a μ is therefore called a scale attribution.

We recall our two main bounds on the propagator

$$G^i_{m,n;k,l} \leqslant K M^{-i} e^{-cM^{-i}(\|m\|+\|n\|+\|k\|+\|l\|)}\ , \tag{4.5}$$

$$\sum_l \max_{n,k} G^i_{m,n;k,l} \leqslant K' M^{-i} e^{-c' M^{-i}\|m\|}\ , \tag{4.6}$$

for some constants K, K' and c, c'.

A considerable fraction of the $4V$ indices initially associated to this graph is determined by external indices of the graph and the δ-function in (4.1). The undetermined indices

are summation indices. Perturbative power counting for a graph consists in finding the indices for which the summation costs a factor M^{2i}, and the ones for which it costs only $\mathcal{O}(1)$, thanks to (4.6). The factor M^{2i} follows from (4.5) after summation over some index[c] $m \in \mathbb{N}^2$,

$$\sum_{m^1,m^2=0}^{\infty} e^{-cM^{-i}(m^1+m^2)} = \frac{1}{(1-e^{-cM^{-i}})^2} = \frac{M^{2i}}{c^2}(1+\mathcal{O}(M^{-i})) \; . \tag{4.7}$$

4.2 Dual graphs

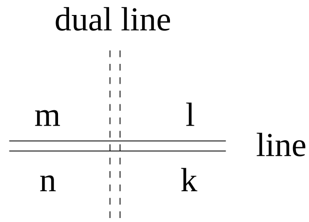
We first want to use as much as possible the δ-functions in (4.1) to reduce the set $\mathcal{I}$ to a truly minimal set $\mathcal{I}'$ of independent indices. It is convenient for this task to consider the dual graph for which the problem becomes analogous to an ordinary problem of momentum routing.

The dual graph of a ribbon graph is obtained by associating to each face a vertex and to each vertex a face. Every line bordering two neighboring faces is replaced by a line joining the two corresponding vertices of the dual graph. Hence, the genus does not change under this duality. We shall write $V' = F$, $F' = V$ for the number of vertices and faces of the dual graph (dual quantities are usually distinguished by a prime). If the initial graph is a ϕ^4-graph, i.e. has coordination 4 at each vertex, we have $4 = I_{f'} + N_{f'}$ for each face $f' \in F'$, where $I_{f'}$ and $N_{f'}$ denote the numbers of edges and external valences, respectively, which belong to f'. The coordination at the vertices of the dual graph is arbitrary.

The construction of the dual of a graph goes as follows: First, for each oriented face of the original ribbon graph, draw an oriented ribbon vertex by assigning

- to a single line of a propagator of the original graph an internal valence of the dual vertex,
- to an external valence of the original graph an external valence of the dual vertex,

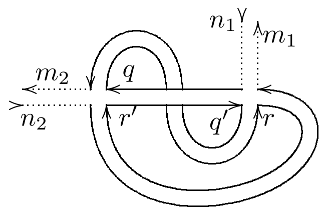
respecting the order according to the arrows on the trajectories. In the second step we connect the valences by the duals of the propagators of the original graph, which is obtained according to



(4.8)

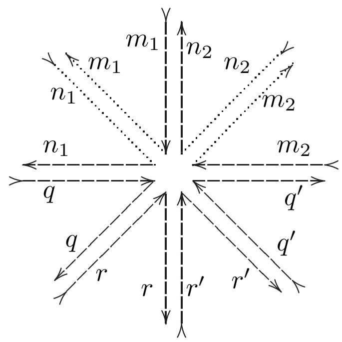
[c]Remember that there are two symplectic pairs, one for spatial dimensions 1 and 2, and the other for spatial dimensions 3 and 4.

Let us consider the following example of a ribbon graph with a single face:



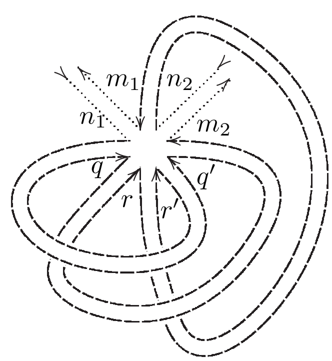
(4.9)

The above rules lead to the following dual vertex:



(4.10)

Now we connect the valences by the duals of the propagators of the original graph, i.e. $n_1 q$ with $r'q'$, qr with $q'm_2$ and $n_2 m_1$ with rr':



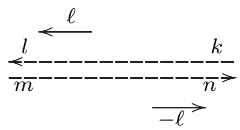
(4.11)

The dual graph is made of the same propagators as the original graph, only the index assignment is different. Whereas in the original graph we have $G_{mn;kl} = \begin{smallmatrix} n & \longleftarrow & k \\ m & \longrightarrow & l \end{smallmatrix}$, the index assignment for propagators in the dual graph is

$$G_{mn;kl} = \begin{smallmatrix} l & \leftarrow\!-\,-\,- & k \\ m & -\,-\,-\!\rightarrow & n \end{smallmatrix}\ . \tag{4.12}$$

The conservation rule $\delta_{m-l,-(k-n)}$ in (4.1) now states that the difference between outgoing and incoming indices of the half-propagator attached to a dual vertex, namely $m-l$, is conserved as minus the corresponding difference $k-n$ at the other end of the propagator. Actually, these index differences describe the *angular momentum*, and the conservation of these differences $\ell = m-l$ and $-\ell = k-n$ is nothing but the angular momentum conservation due to the $SO(2)\times SO(2)$ symmetry of the noncommutative ϕ^4-action. Thus,

taking the incoming indices as the reference, the angular momentum through the dual propagator determines the outgoing indices:



$$m = l + \ell\ , \quad k = n + (-\ell)\ . \tag{4.13}$$

In the same way, there are external angular momenta $\wp$ which enter the dual graph through the external valences. We shall also use the incoming arrow as the reference so that the angular momentum is the index difference between outgoing and incoming arrows. Furthermore, by cyclicity at any vertex, the sum of all incoming differences at a vertex, i.e. the sum of incoming angular momenta, must be zero. Of course, this implies that the total external angular momentum entering a graph is zero, too.

Thus, the angular momentum for the dual graph is exactly like the usual momentum in an ordinary Feynman graph: a momentum that goes out like p at one half-end, must go out as $-p$ at the other half-end, and the total momentum is conserved at any vertex. (In Feynman graphs this follows from translation invariance.) It should be stressed that one has to take into account positivity constraints for the angular momenta ℓ: they lie in $\mathbb{Z}$, but all indices m, n, k, l must be *positive* integers.

We therefore know that the number of *independent* index differences is exactly the number of *loops* L' of the dual graph. For a connected graph, this number is $L' = I - V' + 1$. Furthermore, each index at a vertex is clearly only a function of the differences at a vertex and of a single *reference index* for the dual vertex. If the dual vertex is an external one, we take as the reference index the *outgoing* index at one of the external legs. If the dual vertex is an internal one, we have to make a choice (determined later) for the reference index. These internal vertex reference indices correspond to the loop variable of the original graph. Therefore, after using the conservation rules or δ-functions of each propagator, the number of independent indices to be summed for every graph is simply $V' - B + L' = I + (1 - B)$. Here, $B \geqslant 1$ is the number of boundary components of the original graph, which coincides with the number of external vertices of the dual graph.

Expressing each index in the graph as a function of a set $\mathcal{I}'$ of such independent indices is therefore identical to the problem called momentum routing in a Feynman graph. It is well-known that the solution is not canonical or unique. A good way to root the momenta is to pick a spanning tree $\mathcal{T}_\mu$ of the dual graph, with $V' - 1$ lines, and to use the complement set $\mathcal{L}_\mu$ as the set of fundamental independent differences. The subscript μ refers to the choice of the tree which depends on the scale attribution μ in (4.4).

4.3 Choice of the tree

A given scale attribution $\mu = \{i_\delta\}$ defines an order of lines in the dual graph. We define

$$\delta_1 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_I \qquad \text{if} \qquad i_{\delta_1} \leqslant i_{\delta_2} \leqslant \ldots \leqslant i_{\delta_I}\ . \tag{4.14}$$

In case of equality we make any choice. Let $\delta_1^{\mathcal{T}}$ be the smallest line with respect to this order which is not a tadpole, and $G^{i_{\delta_1^{\mathcal{T}}}}_{m_{\delta_1^{\mathcal{T}}};n_{\delta_1^{\mathcal{T}}};k_{\delta_1^{\mathcal{T}}};l_{\delta_1^{\mathcal{T}}}}$ be the corresponding propagator. The

line $\delta_1^{\mathcal{T}}$ will then connect two vertices $v_1^{\pm}$ and forms the first segment of the tree. We let $\mu_1 := \mu \setminus (\delta_1 \cup \dots \cup \delta_1^T)$ and $\mathcal{T}_1 = \delta_1^{\mathcal{T}} \cup v_1^+ \cup v_1^-$.

In the remaining set μ_1 of lines we identify the smallest line $\delta_2^{\mathcal{T}}$ of μ_1 which does not form a loop when added to $\mathcal{T}_1$. We define $\mu_2 = \mu \setminus (\delta_1 \cup \dots \cup \delta_2^T)$ and

- $\mathcal{T}_2 = \mathcal{T}_1 \cup \delta_2^{\mathcal{T}} \cup v_2^+$ if $\delta_2^{\mathcal{T}}$ connects a vertex v_2^+ to $\mathcal{T}_1$,
- $\mathcal{T}_2 = \mathcal{T}_1 \cup \delta_2^{\mathcal{T}} \cup v_2^+ \cup v_2^-$ if $\delta_2^{\mathcal{T}}$ connects two vertices $v_2^{\pm} \notin \mathcal{T}_1$.

In the n^{th} step, we identify the smallest line $\delta_n^{\mathcal{T}}$ of μ_{n-1} which does not form a loop when added to $\mathcal{T}_{n-1}$. We define $\mu_n = \mu \setminus (\delta_1 \cup \dots \cup \delta_n^T)$ and

- $\mathcal{T}_n = \mathcal{T}_{n-1} \cup \delta_n^{\mathcal{T}} \cup v_n^+$ if $\delta_2^{\mathcal{T}}$ connects a vertex v_n^+ to $\mathcal{T}_{n-1}$,
- $\mathcal{T}_n = \mathcal{T}_{n-1} \cup \delta_n^{\mathcal{T}} \cup v_n^+ \cup v_n^-$ if $\delta_2^{\mathcal{T}}$ connects two vertices $v_n^{\pm} \notin \mathcal{T}_{n-1}$,
- $\mathcal{T}_n = \mathcal{T}_{n-1} \cup \delta_n^{\mathcal{T}}$ if $\delta_2^{\mathcal{T}}$ connects two disjoint subsets of $\mathcal{T}_{n-1}$.

The $(V'-1)^{\text{th}}$ step leads to the desired tree $\mathcal{T}_\mu = \mathcal{T}_{V'-1}$. The importance of this construction is the fact that any line $\delta_j^{\mathcal{L}} \in \mathcal{L}_\mu$ which connects *different* vertices $v_j^{\pm}$ of the tree has a scale index $i_{\delta_j^{\mathcal{L}}}$ which is *not smaller* than any scale index of the lines *in the tree* which connect $v_j^{\pm}$. Such a tree optimization for a given scale attribution is one of the most basic tools of constructive field theory [10], so it is an encouraging sign that it appears also here in a natural way.

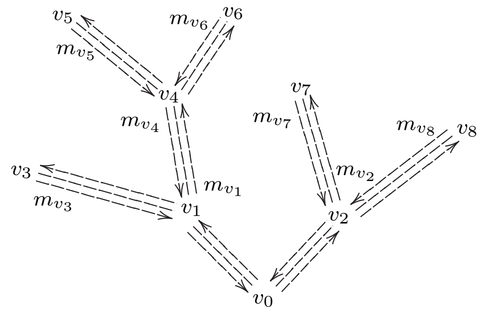
In the graphical representations we distinguish the tree by triple dashed lines.

4.4 Index assignment for a tree

For the previously constructed tree T_μ we select one of its $B \geqslant 1$ external vertices as the root v_0 of the tree. We relabel the vertices in the tree such that all vertices in the subtree above a vertex v_n have a label bigger than n.

The order (4.14) of the lines of the graph provides us with a convenient position for the main reference index m at each vertex. If v is an internal vertex, we let δ_v be the smallest line in (4.14) which is attached to v. By construction of the tree we know that either δ_v is a tadpole, or it belongs to the tree. We choose the *outgoing* index (without the arrow when viewed from the vertex) of this particular line δ_v as the main reference index m_v. We let $\mathcal{G}_{\mathcal{M}}$ be the set of lines at which a main reference index resides. It is possible that both outgoing indices of a line $\delta_v = \delta_{v'}$ attached to v and v' are main reference indices. In this case we let δ_v appear twice in $\mathcal{G}_{\mathcal{M}}$. Thus, $\mathcal{G}_{\mathcal{M}}$ consists of $V' - B$ elements. If v is an external vertex, we take as the "main reference index" the outgoing index at any external leg. The following graph shows a typical situation of a tree and its

main reference indices, assuming absence of tadpoles and $B = 1$:

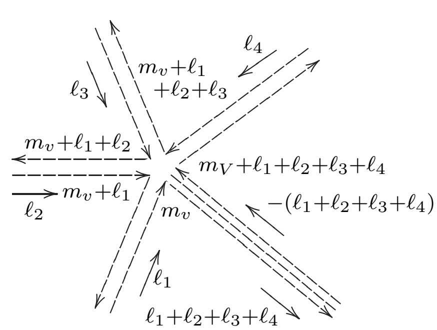


(4.15)

One can now write down every index in a unique way in terms of

- $V' - B$ main reference indices m_v,
- L' internal angular momenta ℓ_δ for the set $\mathcal{L}_\mu$ of "loop lines"
- B main reference indices at external vertices,
- N external angular momenta $\wp_\varepsilon$ for the set $\mathcal{N}$ of external lines.

The rule goes as follows. One writes first the indices for all the "leaves" of the tree, that is the vertices (distinct from the root) with coordination 1 in the tree (i.e. vertices v_3, v_5, v_6, v_7 and v_8 in (4.15)). For them, starting from the main index m_v (at the left of the unique line of $\mathcal{T}_\mu$ at v that goes towards the root, unless a tadpole at v has the smallest scale), which agrees with the incoming index of the next line at v in clockwise direction, we compute all other indices by turning *clockwise* around the vertex and by adding the angular momenta (internal or external ones) associated according to (4.13) with the loop lines $\delta_1, \dots, \delta_k$ and possibly external lines $\varepsilon_1, \dots, \varepsilon_{k'}$. This gives indices $m_v + \ell_1$, $m_v + \ell_1 + \ell_2$, ...until we arrive at $m_v + \ell_1 + \dots + \ell_{k+k'}$ which is at the right of the unique line at v that goes towards the root. Some of the ℓ_j could be external angular momenta. Then we can compute the angular momentum associated to that line: it is simply $-(\ell_1 + \dots + \ell_{k+k'})$. The following picture shows these assignments for a particular "leaf":



(4.16)

Having done this for all leaves, we can prune these leaves and consider the next layer of vertices down towards the root (i.e. vertices v_2 and v_4 in (4.15)) and reiterate the argument.

Any summation index at a vertex v is now clearly equal to m_v plus a linear combination of angular momenta ℓ_δ for the set of lines $\mathcal{L}_v \cup \mathcal{N}_v$ hooked to the "subtree above v", that is the lines hooked at least at one end to a vertex v' such that the unique path from v' to v_0 passes through v.

We split therefore the set $\mathcal{I}'$ of independent indices to be summed into the two sets:

- the set $\mathcal{M}_\mu = \{m_v\}$ of main reference indices at inner vertices, which consists of $V' - B$ elements,
- the set $\mathcal{J}_\mu = \{\ell_\delta,\ \delta \in \mathcal{L}\}$ of angular momenta, which consists of L' elements.

The amplitude of the graph G is now written as:

$$A_G = \sum_\mu \sum_{\mathcal{M}_\mu, \mathcal{J}_\mu} \prod_{\delta \in G} G^{i_\delta}_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}\ \chi(\mathcal{M}_\mu, \mathcal{J}_\mu)\ , \tag{4.17}$$

where the sum over $\mathcal{M}_\mu$ is over positive integers, but the sum over $\mathcal{J}_\mu$ is over relative integers and the function $\chi(\mathcal{M}_\mu, \mathcal{J}_\mu)$ is the characteristic function which states that all the functions $\{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)\}$ are positive. The dependence of A_G on the external indices (B reference indices at external vertices and N external angular momenta) is not made explicit.

4.5 Power-counting

Our goal is now to prove that sums over difference indices can always be performed through (4.6), hence at no cost, *using precisely the propagator G^{i_δ} to perform the sum over the angular momentum ℓ_δ*. However, as the angular momenta are entangled, we need appropriate maximizations of the other $G^{i'_\delta}$ over ℓ_δ. These maximizations require a carefully chosen order. For processing the angular momenta, all main reference indices $\mathcal{M}_\mu$ and the external indices are kept constant.

We start with the highest labelled leaf $v = v_{V'-1}$ according to the previously defined order of vertices. Let this vertex carry loop lines with corresponding index assignments according to (4.16). We first assume that δ_4 is not a tadpole. Then, we should sum over ℓ_1, ℓ_2, ℓ_3 *after* the ℓ_4-summation, i.e. ℓ_1, ℓ_2, ℓ_3 are constant with respect to the ℓ_4-summation. This guarantees that at least one side of the lines $\delta_1, \delta_2, \delta_3$, namely the side attached to v, is independent of ℓ_4. We would thus maximise the lines in the tree over ℓ_4, but possibly also the ℓ_4-dependence of the other ends of $\delta_1, \delta_2, \delta_3$. Looking e.g. at δ_3 which connects v with v', the corresponding propagator would be

$$G^{i_3}_{m_v+\ell_1+\ell_2, k_{v'}-\ell_3; k_{v'}, m_v+\ell_1+\ell_2+\ell_3}\ , \tag{4.18}$$

see (4.12). Note that the incoming index viewed from v' is always the reference index, labelled $k_{v'}$ in this case, and the corresponding outgoing index is obtained by adding the opposite angular momentum $-\ell_3$. The reference index $k_{v'}$ is either independent of ℓ_4 or $k_{v'} = k^0_{v'} \pm \ell_4$ for $k^0_{v'}$ being independent of ℓ_4. Thus, the maximization of G^{i_3} over ℓ_4 and

possibly over other angular momenta ℓ_j, $j > 4$, is precisely of the structure in (4.6). Later, it will be essential that concerning the ℓ_3-summation to be applied to G^{i_3} we keep ℓ_2 and ℓ_1 constant, because $k_{v'}$ might also depend on ℓ_1 and/or ℓ_2. After having maximized all other propagators over ℓ_4 we restrict the ℓ_4-summation to the line δ_4, which connects v to v'' and corresponds to the propagator

$$G^{i_4}_{m_v+\ell_1+\ell_2+\ell_3,k_{v''}-\ell_4;k_{v''},m_v+\ell_1+\ell_2+\ell_3+\ell_4} \; . \tag{4.19}$$

Here, there might be previous maximizations of $k_{v''}$ over ℓ_j, $j > 4$, which we can bound by the maximization of (4.19) over all $k_{v''}$. To that maximized propagator we apply for given ℓ_1, ℓ_2, ℓ_3 the ℓ_4-summation. Later, when applying in the same manner the ℓ_j-summations, $j = 1, 2, 3$, to G^{i_j}, we have to maximise the previously processed G^{i_4} over ℓ_1, ℓ_2, ℓ_3. In other words, the line δ_4 and the ℓ_4-summation taken at the correct place yield, see (4.6),

$$\max_{\ell_1,\ell_2,\ell_3} \left(\sum_{\ell_4} \left(\max_{\ell_j,\, j>4} G^{i_4}_{m_v+\ell_1+\ell_2+\ell_3,k_{v''}(\ell_j)-\ell_4;k_{v''}(\ell_j),m_v+\ell_1+\ell_2+\ell_3+\ell_4} \right) \right) \leqslant K M^{-i_4} \; . \tag{4.20}$$

To summarize, we can estimate the ℓ_j-summations restricted to the line δ_j *at the highest-labelled leaf* by (4.6) if all angular momenta ℓ_k associated with those lines δ_k which are encountered *before* δ_j when turning *clockwise* around the leaf (starting from the main reference index) are kept constant. It is not difficult to convince oneself that the same rule also holds for tadpoles.

Now we remove the highest-labelled vertex $v_{V'-1}$ and look at the highest labelled leaf $v_{V'-2}$ of the reduced tree $\mathcal{T} \setminus v_{V'-1}$. Let it be again represented by (4.16). We continue to label the lines at $v_{V'-2}$ in clockwise order from the distinguished main reference index. Lines which belong to $\mathcal{T}$ and lines attached to $v_{V'-1}$ are left out, because their angular momenta are already identified. These previous angular momenta can be considered as *fixed* external ones which are summed *after* the new angular momenta at $v_{V'-2}$. Clearly, the same rule as above, namely to sum *later* over those angular momenta which are encountered *earlier* in clockwise order, allows us to use (4.6) for the summation over the new angular momenta at $v_{V'-2}$.

We repeat this procedure until we arrive at the root v_0. The result is a bound for the $\mathcal{J}_\mu$-summation in (4.17). There, all propagators which correspond to tree-lines $\delta \in \mathcal{T}_\mu$ are maximized over $\mathcal{J}_\mu$. For tree lines where all indices depend on $\mathcal{J}_\mu$ the bound, due (4.5), is given by

$$\max_{\mathcal{J}_\mu} G^{i_\delta}_{m_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu),n_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu);k_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu),l_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu)} \leqslant K M^{-i_\delta} \;, \qquad \delta \in \mathcal{T}_\mu \;, \quad \delta \notin \mathcal{G}_\mu \; . \tag{4.21}$$

If one of the indices of $\delta \in \mathcal{T}_\mu$ is a main reference index at v, we have

$$\max_{\mathcal{J}_\mu} G^{i_\delta}_{m_v,n_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu);k_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu),l_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu)} \leqslant K M^{-i_\delta} e^{-cM^{-i_\delta}\|m_v\|} \;, \qquad \delta \in \mathcal{T}_\mu \;, \quad \delta \in \mathcal{G}_\mu \; . \tag{4.22}$$

If two indices of δ are main reference indices at v, v', we have

$$\max_{\mathcal{J}_\mu} G^{i_\delta}_{m_v,n_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu);m_{v'},l_\delta(\mathcal{M}_\mu,\mathcal{J}_\mu)} \leqslant KM^{-i_\delta} e^{-cM^{-i_\delta}(\|m_v\|+\|m_v{}'\|)} , \qquad \delta \in \mathcal{T}_\mu \;, \quad \delta \in \mathcal{G}_\mu \setminus \delta \;. \tag{4.23}$$

Next, each summed propagator which corresponds to a line $\delta \in \mathcal{L}_\mu$ delivers according to (4.20) a factor KM^{-i_δ},

$$\max_{\ell_1,\dots,\ell_{\delta-1}} \left(\sum_{\ell_\delta} \left(\max_{\ell_{\delta+1},\dots,\ell_{L'}} G^{i_\delta}_{m_\delta(\mathcal{J}_\mu\setminus\ell_\delta),n_\delta(\mathcal{J}_\mu\setminus\ell_\delta)-\ell_\delta;k_\delta(\mathcal{J}_\mu\setminus\ell_\delta),l_\delta(\mathcal{J}_\mu\setminus\ell_\delta)+\ell_\delta} \right)\right) \leqslant K'M^{-i_\delta} \;, \qquad \delta \in \mathcal{L}_\mu \;, \;\; \delta \notin \mathcal{G}_\mu \;. \tag{4.24}$$

In the case that $\delta_j \in \mathcal{L}_\mu$ is a tadpole at v which has the smallest scale index among the set of lines at v we obtain from (4.6) the bound

$$\max_{\ell_1,\dots,\ell_{\delta-1}} \left(\sum_{\ell_\delta} \left(\max_{\ell_{\delta+1},\dots,\ell_{L'}} G^{i_\delta}_{m_v,n_\delta(\mathcal{J}_\mu\setminus\ell_\delta)-\ell_\delta;k_\delta(\mathcal{J}_\mu\setminus\ell_\delta),l_\delta(\mathcal{J}_\mu\setminus\ell_\delta)+\ell_\delta} \right)\right) \leqslant K'M^{-i_\delta}\, e^{-c'M^{-i_\delta}\|m_v\|} \;, \qquad \delta \in \mathcal{L}_\mu \;, \;\; \delta \in \mathcal{G}_\mu \;. \tag{4.25}$$

Eventually, there will be indices $m_\varepsilon, n_\varepsilon$ which are fixed as external ones. Each one delivers according to (4.5) an additional factor $e^{-cM^{-i_\varepsilon}\|m_\varepsilon\|}$ and $e^{-cM^{-i_\varepsilon}\|n_\varepsilon\|}$, respectively, because these decays cannot be removed by maximizing loop momenta. For external indices which are not connected to internal lines we put $c \equiv 0$.

Altogether, the $\mathcal{J}_\mu$-summation in (4.17) can be estimated by

$$\begin{aligned} A_G \leqslant \sum_\mu \sum_{m_1,\dots,m_{V'-B}\in\mathbb{N}^2} &\left(\prod_{\delta'\in\mathcal{G}_\mu} e^{-cM^{-i_{\delta'}}\|m_{v(\delta')}\|} \right) \left(\prod_{\delta\in G} KM^{-i_\delta} \right) \\ &\times \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon}\|m_\varepsilon\|} \right) \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon}\|n_\varepsilon\|} \right) \;, \end{aligned} \tag{4.26}$$

where $m_{v(\delta')}$ is the main reference index at $\delta' \in \mathcal{G}_\mu$. After summation over $m_1, \dots, m_{V'-B}$ we have

$$A_G \leqslant \sum_\mu \frac{K^I}{c^{2(V'-B)}} \Big(M^{-\sum_{\delta\in G} i_\delta} \Big) \Big(M^{2\sum_{\delta'\in\mathcal{G}_\mu} i_{\delta'}} \Big) \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon}\|m_\varepsilon\|} \right) \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon}\|n_\varepsilon\|} \right) . \tag{4.27}$$

The dangerous region of the sum over the scale attribution is at large scale indices. To identify this region, we associate to the order (4.14) of lines a sequence of subgraphs $G_I \subset G_{I-1} \subset \cdots \subset G_1 = G$ of the original ribbon graph by defining G_γ as the possibly disconnected set of lines $\delta_{\gamma'} \geqslant \delta_\gamma$, together with all vertices attached to them. To G_γ

we associate the scale attribution μ_γ which starts from an irrelevant low-scale cut-off $i_{\gamma-1} \leqslant i_{\gamma'}$. We conclude from (4.27) that the amplitude A_{G_γ} corresponding to the subgraph G_γ diverges if

$$\omega_\gamma := 2(V'_\gamma - B_\gamma) - I_\gamma = 2F_\gamma - 2B_\gamma - I_\gamma = (2 - \tfrac{N_\gamma}{2}) - 2(2g_\gamma + B_\gamma - 1) \tag{4.28}$$

is non-negative, where N_γ, V_γ, $I_\gamma = I - \gamma + 1$, F_γ and B_γ are the numbers of external legs, vertices, edges, faces and external faces of G_γ, respectively, and $g_\gamma = 1 - \frac{1}{2}(V_\gamma - I_\gamma + F_\gamma)$ is its genus. We have thus proven the following

Theorem 6 *The sum over the scale attribution μ in (4.27) converges if for all subgraphs $G_\gamma \subset G$ we have $\omega_\gamma < 0$.*

For the total graph $\gamma = G$ the power-counting degree becomes $\omega = (2-\frac{N}{2})-2(2g+B-1)$, which reproduces the power-counting degree derived in [8].

We consider in Appendix B a few examples for the sum over the scale attribution.

4.6 Subtraction procedure for divergent subgraphs

The power-counting theorem 6 implies that planar subgraphs with two or four external legs are the only ones for which the sum over the scale attribution can be divergent. These graphs require a separate analysis. We first see from Theorem 2 that

- only those planar four-leg subgraphs with *constant index* along the trajectory are marginal,
- only those planar two-leg graphs with *constant index* along the trajectory are relevant,
- only those planar two-leg graphs with an *accumulated index jump of 2* along the trajectory are marginal.

For the other types of graphs there is a sufficient power of M^{-i} through the terms $(M^{-i}l)^\delta$ in (3.33) which makes the sum over the scale attribution convergent.

For the remaining truly divergent graphs one performs similarly as in the BPHZ scheme a Taylor subtraction about vanishing external indices. For instance, a marginal four-leg graph with amplitude $A_{mn;nk;kl;lm}$ is written as

$$A_{mn;nk;kl;lm} = (A_{mn;nk;kl;lm} - A_{00;00;00;00}) + A_{00;00;00;00}\ . \tag{4.29}$$

The difference of graphs $A_{mn;nk;kl;lm} - A_{00;00;00;00}$ can be expressed as a linear combination involving the *composite propagators* (3.54). See also [5] for more details. Then, the estimation (3.57) provides an additional factor M^{-i} which makes the sum over the scale attribution for the difference $A_{mn;nk;kl;lm} - A_{00;00;00;00}$ convergent.

Eventually, there remain only the four divergent base functions $A_{\substack{00\\00};\substack{00\\00};\substack{00\\00};\substack{00\\00}}$, $A_{\substack{00\\00};\substack{00\\00}}$, $(A_{\substack{10\\00};\substack{01\\00}} - A_{\substack{00\\00};\substack{00\\00}})$ and $A_{\substack{11\\00};\substack{00\\00}}$, taking into account the symmetry properties of the model.

These are normalized to their "experimentally" determined values: the physical coupling constant, the physical mass, the physical field amplitude and the physical frequency of the harmonic oscillator potential, respectively.

At the end, any graph appearing in the noncommutative ϕ^4-model has an amplitude which is uniquely expressed by four normalization conditions as well as convergent sums over the scale attribution. Thus, the model is renormalizable to all orders.

5 Conclusion

For many years, noncommutative quantum field theories were supposed to be ill-behaved due to the UV/IR-mixing problem [3]. Meanwhile, it turned out [8, 5] that at least the Euclidean noncommutative ϕ^4_4-model is as good as its commutative version: it is renormalizable to all orders. In fact, the noncommutative ϕ^4_4-model is even better than the commutative version with respect to one important issue: the behavior of the β-function.

It is well-known that the main obstacle to a rigorous construction of the commutative ϕ^4_4-model is the non-asymptotic freedom of the theory. The noncommutative model is very different: The computation of the β-function [11] shows that the ratio of the bare coupling constant to the square of the bare frequency parameter remains (at the one loop level) constant over all scales, $\frac{\lambda}{\Omega^2} = \text{const}$. (This was noticed in [13].) As the bare frequency is bounded by 1, this means that the bare coupling constant is *bounded*. For appropriate renormalized values, the coupling constant can be kept arbitrarily small throughout the renormalization flow. We are, therefore, optimistic that a rigorous construction of the noncommutative ϕ^4_4-model will be possible.

In this paper we have undertaken the first important steps in this direction. We have formulated the perturbative renormalization proof in a language which admits a direct extension to constructive methods. More details about our program are given in [14]. Moreover, our new renormalization proof is much more efficient than the previous one (by a factor of 3 when looking at the number of pages). Eventually, we have established analytical bounds for the asymptotic behavior of the propagator which before were only established numerically.

A The case $\Omega = 1$

Our proofs do not apply to the case $\Omega = 1$, because the scale parameter becomes $M = 1$, see (3.11) and (3.10). However, the case $\Omega = 1$ can be directly treated. According to (3.2), only the terms with $u = m = l$ survive:

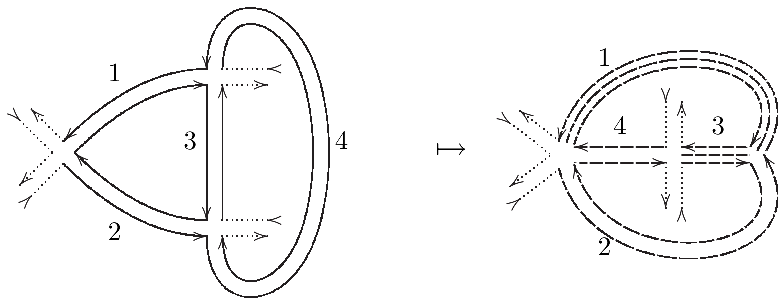
$$G^{\Omega=1}_{mn;kl} = \frac{\theta}{8}\int_0^1 d\alpha (1-\alpha)^{\frac{\mu_0^2\theta}{8}+(\frac{D}{4}-1)+\frac{1}{2}(\|m\|+\|k\|)}\,\delta_{ml}\delta_{nk} \tag{A.1}$$

$$= \frac{\delta_{ml}\delta_{nk}}{\mu_0^2 + \frac{2}{\theta}(\|m\|+\|n\|+\|k\|+\|l\|+\frac{D}{2})}\,. \tag{A.2}$$

The exponential decay of the propagator in any index is easily obtained from (A.1) for all slices. Moreover, the l-sum is trivial to perform due to the index conservation δ_{ml} at each trajectory.

B Examples for the sum over the scale attribution

We consider a few examples which underline the relation between (4.27) and divergent subgraphs. The first one is:



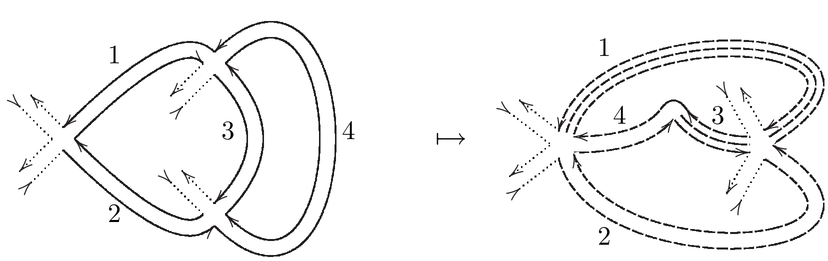
(B.1)

The tree in the dual graph corresponds to $i_1 \leqslant i_2$ and $i_3 \leqslant i_4$ and $i_1 \leqslant i_4$. For the particular choice $i_1 \leqslant i_2 \leqslant i_3 \leq i_4$ of the scale attribution we have to compute according to (4.27) the following sum, taking into account that the volume factor from the summation over the main reference index is associated to i_1. We put $x = M^{-1}$, $0 < x < 1$:

$$\begin{aligned}
\sum_{i_4=0}^{\infty}\sum_{i_3=0}^{i_4}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2} x^{-i_1+i_2+i_3+i_4} &= \sum_{i_4=0}^{\infty}\sum_{i_3=0}^{i_4}\sum_{i_2=0}^{i_3} \frac{1-x^{i_2+1}}{1-x} x^{i_3+i_4} \\
&= \sum_{i_4=0}^{\infty}\sum_{i_3=0}^{i_4} \Big(\frac{x^{i_3}-2x^{i_3+1}}{(1-x)^2} + i_3\frac{x^{i_3}}{1-x} + \frac{x^{2i_3+2}}{(1-x)^2}\Big) x^{i_4} \\
&= \sum_{i_4=0}^{\infty} \Big(\frac{x^{i_4}-2x^{2i_4+1}+2x^{2i_4+3}-x^{3i_4+4}}{(1-x)^3(1+x)} - \frac{i_4\, x^{2i_4+1}}{(1-x)^2}\Big) \\
&= \frac{1}{(1-x)^2(1-x^2)(1-x^3)}\;. \qquad \text{(B.2)}
\end{aligned}$$

One can check that any order of scales i_j leads to a convergent sum.

On the other hand,



(B.3)

The tree in the dual graph corresponds to $i_1 \leqslant i_2$ and $i_3 \leq i_4$ and $i_1 \leqslant i_4$. For the particular choice $i_1 \leqslant i_2 \leqslant i_3 \leqslant i_4$ of the scale attribution we have to compute according to (4.27) the following sum, taking into account that the volume factor from the summation over the main reference index is associated to i_3:

$$\begin{aligned}
\sum_{i_4=0}^{\Lambda}\sum_{i_3=0}^{i_4}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2} x^{i_1+i_2-i_3+i_4} &= \sum_{i_4=0}^{\infty}\sum_{i_3=0}^{i_4}\sum_{i_2=0}^{i_3} \frac{x^{i_2}-x^{2i_2+1}}{1-x} x^{-i_3+i_4} \\
&= \sum_{i_4=0}^{\Lambda}\sum_{i_3=0}^{i_4} \Big(\frac{x^{-i_3}-x(1+x)+x^{3+i_3}}{(1-x)^2(1+x)}\Big) x^{i_4} \\
&= \sum_{i_4=0}^{\Lambda}\sum_{i_3=0}^{i_4} \Big(\frac{1-2x^{i_4+1}+2x^{i_4+3}-x^{2i_4+4}}{(1-x)^3(1+x)} - \frac{i_4\, x^{i_4+1}}{(1-x)^2}\Big) \\
&= \frac{\Lambda}{(1-x)^2(1-x^2)} + \frac{1-2x+4x^2}{(1-x)^2(1-x^2)^2} \\
&+ \frac{\Lambda\, x^{\Lambda+2}}{(1-x)^3} + \frac{3x^{\Lambda+2}+4x^{\Lambda+3}-x^{\Lambda+4}-2x^{\Lambda+5}+x^{2\Lambda+6}}{(1-x)^2(1-x^2)^2}\;. \qquad \text{(B.4)}
\end{aligned}$$

Thus, although the graph is *superficially convergent*, there is a *subdivergence* given by the subgraph made of propagators 3 and 4, which leads to a divergent sum over the scale attribution. Therefore, the subdivergence must be treated first by Taylor subtraction at vanishing momenta in the same way as in [5].

References

[1] V. Schomerus, "D-branes and deformation quantization," JHEP **9906** (1999) 030 [arXiv:hep-th/9903205].

[2] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].

[3] S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].

[4] I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and renormalization," JHEP **0103** (2001) 001 [arXiv:hep-th/0008090].

[5] H. Grosse and R. Wulkenhaar, "Renormalization of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," arXiv:hep-th/0401128, to appear in Commun. Math. Phys.

[6] K. G. Wilson and J. B. Kogut, "The renormalization group and the epsilon expansion," Phys. Rept. **12** (1974) 75.

[7] J. Polchinski, "Renormalization and effective Lagrangians," Nucl. Phys. B **231** (1984) 269.

[8] H. Grosse and R. Wulkenhaar, "Power-counting theorem for non-local matrix models and renormalization," arXiv:hep-th/0305066, to appear in Commun. Math. Phys.

[9] E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," Phys. Lett. B **533** (2002) 168 [arXiv:hep-th/0202039].

[10] V. Rivasseau, "From perturbative to constructive renormalization," Princeton Univ. Press, Princeton, 1991.

[11] H. Grosse and R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," Eur. Phys. J. C **35** (2004) 277 [arXiv:hep-th/0402093].

[12] E. Langmann, R. J. Szabo and K. Zarembo, "Exact solution of quantum field theory on noncommutative phase spaces," JHEP **0401** (2004) 017 [arXiv:hep-th/0308043].

[13] H. Grosse and R. Wulkenhaar, "Renormalization of ϕ^4 theory on noncommutative $\mathbb{R}^4$ to all orders," arXiv:hep-th/0403232, to appear in Lett. Math. Phys.

[14] V. Rivasseau and F. Vignes-Tourneret, "Non-commutative renormalization," arXiv:hep-th/0409312.